Modelling and control summaries

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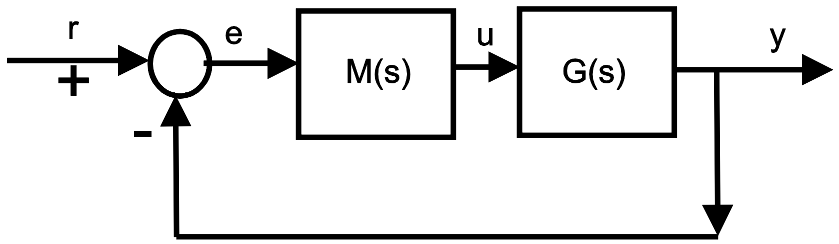
Root-loci 1: Introduction

This brief summary assumes readers are familiar with the concept of feedback and block diagrams. For simplicity the focus is on the simplest form of block diagram, although it is relatively straightforward to extend the concepts to more complex arrangements. Hence consider a process *G(s)* and a compensator *M(s)* which is expressed as a gain K multiplied by a transfer function. The closed-loop transfer function is $G_c(s)$.



$$M(s) = K\tilde{M}(s)$$

$$G_c = \frac{GK\tilde{M}}{1+GK\tilde{M}}$$

What is a root?
In the context of feedback control and root-loci, a root is a closed-loop pole, that is a root of $p_c(s)$.

$$G_c = \frac{GK\tilde{M}}{1+GK\tilde{M}} = \frac{n}{p_c}$$

What is/are a root-loci?
As the compensator gain K is changed, the pole-polynomial $p_c(s)$ changes. Root-loci describe how the closed-loop poles ρ change, viewed in the complex plane.

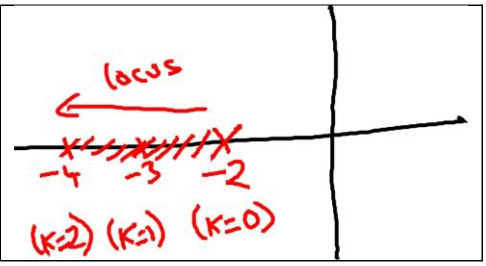
- **Root-loci are the paths followed by the closed-loop poles as K changes.**
- **These are important because poles tell us about stability and behaviour.**

EXAMPLE 1: Find the root-loci for the following compensator and process pair.

$$\left\{ G(s) = \frac{1}{s+2}; \quad M(s) = K \right\} \Rightarrow \quad G_c(s) = \frac{K}{s+2+K}; \quad \rho = -2-K$$

K	0	1	2	3	4	5	6
Pole	-2	-3	-4	-5	-6	-7	-8

Compute the closed-loop poles for different values of K and plot these on a complex plane. The locus (for positive K), begins at -2 and progresses monotonically towards minus infinity as K increases.



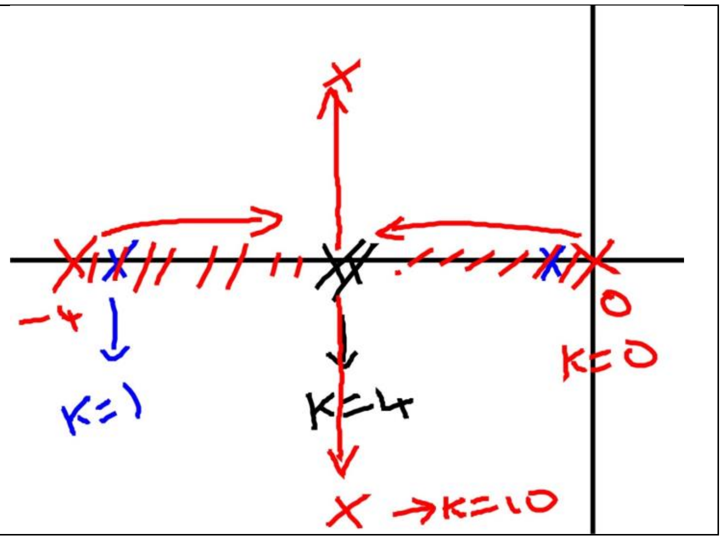
EXAMPLE 2: Find the root-loci for the following compensator and process pair.

$$\left\{ G(s) = \frac{1}{s^2 + 4s}; \quad M(s) = K \right\} \Rightarrow G_c(s) = \frac{K}{s^2 + 4s + K}; \quad \rho = -2 \pm \sqrt{4 - K}$$

K	0	1	4	10	50
Poles	0, -4	-0.3, -3.7	-2, -2	-2+j2.45 -2-j2.45	-2+j6.8 -2-j6.8

Compute the closed-loop poles for different values of K and plot these on a complex plane.

- The loci (for positive K), begin at 0 and -4.
- Both parts move towards -2 where they meet when K=4. For K>4, the poles have an ever increasing imaginary part, but a fixed real part.



Summary and exercises

1. A root-loci can be computed using simple calculation of the closed-loop poles for all positive values of compensator gain K and plotting the results on an Argand diagram.
2. This is somewhat tedious, but readers are advised to go through the process for a few examples to ensure they understand the concept fully.

Define the closed-loop pole polynomial for the following pairs:

$$\left\{ \begin{array}{l} G = \dfrac{1}{s^2 + 3s + 2} \\ M(s) = \dfrac{0.4}{s} K \end{array} \right\}; \quad \left\{ \begin{array}{l} G = \dfrac{1}{s^3 + 3s^2 + 3s + 1} \\ M(s) = \dfrac{0.2(s+3)}{(s+4)} K \end{array} \right\}; \quad \left\{ \begin{array}{l} G = \dfrac{1}{s^3 + 6s^2 + 11s + 6} \\ M(s) = \dfrac{0.2}{s} K \end{array} \right\}$$

Validate the root-loci for the following system compensator pairs:

$G(s) = \dfrac{s+1}{(s+2)(s+3)}; \quad M(s) = K$	$G(s) = \dfrac{s-2}{s(s+2)(s+4)}; \quad M(s) = K$