Modelling and control summaries

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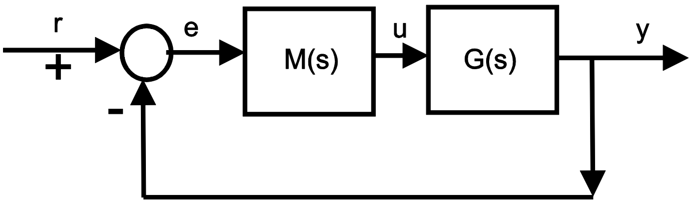
Root-loci 17: Angles of arrival/departure

The focus is on the simplest form of block diagram, a process *G(s)* and a compensator *M(s)* which is expressed as a gain K multiplied by a transfer function. The closed-loop transfer function is $G_c(s)$.



$$M(s) = K\tilde{M}(s)$$

$$G_c = \frac{GK\tilde{M}}{1+GK\tilde{M}} = \frac{Kn}{Kn+d}$$

Conditions for closed-loop poles

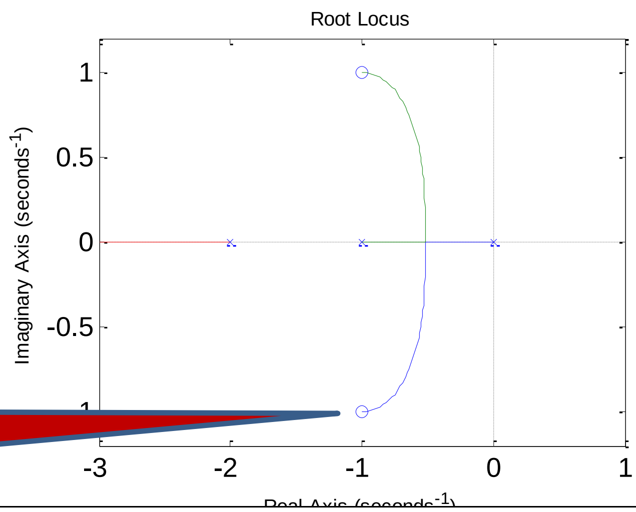
$$p_c = 0 \quad \equiv \quad \left\{ \begin{array}{c} Kn + d = 0 \\ K\dfrac{n}{d} = -1 \\ \arg\left(\dfrac{n}{d}\right) = \pm 180^o \end{array} \right\}$$

Complex poles or zeros

When a system has complex conjugate pairs of poles or zeros, these serve as departure or arrival points for the loci.

A key question **linked to design** is to determine from what direction the loci approach these positions.

At what angle does loci approach this zero?



How to determine angles of arrival and departure

Use the angle criteria for a candidate value 's' near the relevant pole/zero, that is:

$$\angle n - \angle d = \pm 180^o$$

$$n = (s-z_1)(s-z_2)\cdots(s-z_m) \quad \Rightarrow \quad \angle n = \sum \angle(s-z_i)$$

$$d = (s-p_1)(s-p_2)\cdots(s-p_k) \quad \Rightarrow \quad \angle d = \sum \angle(s-p_i)$$

$$\sum \angle(s-z_i) - \sum \angle(s-p_i) = \pm 180^o$$

Using approximation

If s (on loci) is very close to z_i, then one can approximate most of the angles by writing $s=z_i$.

$$\sum_{j\neq i}\angle(z_i - z_j) + \angle s - z_i - \sum \angle(z_i - p_j) = \pm 180^o$$

Rearranging the above equation we find the angle or arrival/departure directly.

$$\angle s - z_i = -\sum_{j\neq i}\angle(z_i - z_j) + \sum \angle(z_i - p_j) \pm 180^o$$

Example of using approximation

$$G = \frac{(s+1+j)(s+1-j)}{s(s+2)(s+1)}$$

$$z_1 = -1+j,\ z_2 = -1-j$$

$$p_1 = 0,\ p_2 = -1,\ p_3 = -2$$

Find angle of arrival at z_1.

Using formulae above, substitute in z_i, p_i:

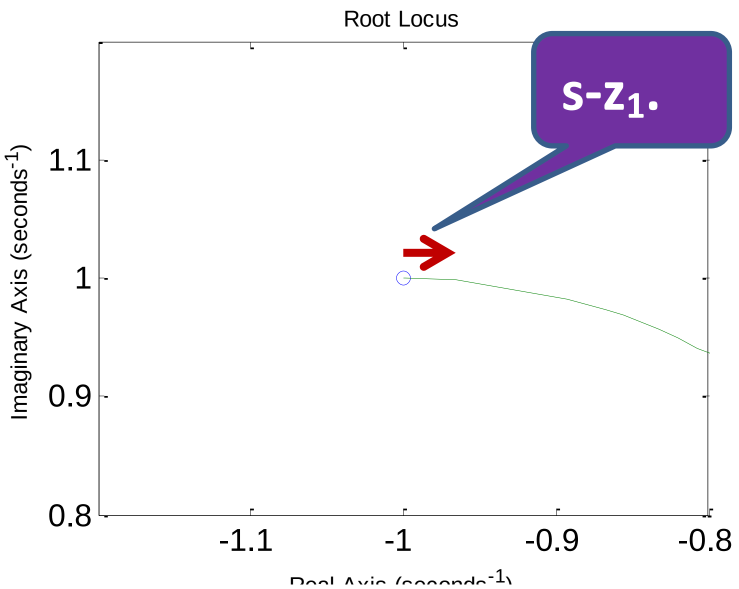
$$\angle s - z_1 = -\angle[(-1-j)-(-1+j)] + \angle[(-1-j)-(0)] + \angle[(-1-j)-(-1)] + \angle[(-1-j)-(-2)] \pm 180^o$$

Hence:

$$\angle s - z_1 = -\angle -2j + \angle(-1-j) + \angle -j + \angle(1-j) \pm 180^o$$

$$\angle s - z_1 = 90 + (-135) - 90 - 45 \pm 180^o = 0^o$$

A blow up of the root locus near z_1 shows that the loci does indeed approach in the direction given, that is 0 degrees!



REMARKS: Calculating angles of arrival and departure is quite tedious by hand and in the days of modern computing it would be rare to do this. However, understanding the procedure can give some useful insight that helps with design.