Modelling and control summaries

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Offset 1 – signal steady-state

Focus of this summary is to consider what type of signal has a finite asymptotic value, that is:
$y_\infty = \lim_{t\to\infty} y(t)$ **and moreover y^∞ is bounded.**
Secondly, can we classify those asymptotic values in a meaningful way.

WHICH OF THESE SIGNALS HAVE A BOUNDED AYSMPTOTIC VALUE?

$$e^{0.04t}, \quad e^{0.04t}\sin 2t, \quad 0.2t,$$

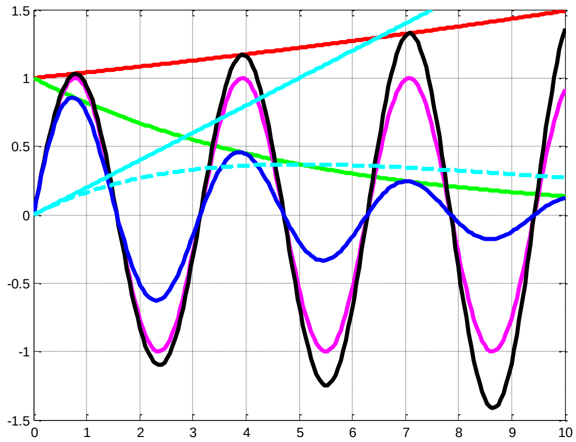
divergent

$$e^{-0.2t}, \quad e^{-0.2t}\sin 2t, \quad 0.2te^{-0.2t}$$

convergent

oscillatory

$$\sin 2t$$



1. A pure sinusoid does not have an asymptotic value ALTHOUGH IT IS BOUNDED.
2. An exponential with a positive exponent does not have an asymptotic value (divergent).
3. Positive powers of time ($t, t^2, \ldots$) do not have asymptotic values (divergent).

ONLY constants and signals multiplied by an exponential with a negative exponent converge to a fixed value. HEREAFTER CONSIDER ONLY CONVERGENT SIGNALS.

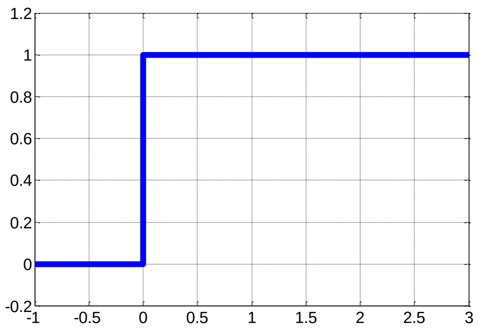
What is the asymptotic value for the following convergent signals?				
e^{-4t} ➔ 0	$e^{-0.2t}$ ➔ 0	$e^{-0.05t}\sin 2t$ ➔ 0	$3+2t\,e^{-0.001t}$ ➔ 3+0	$-2+2t\,e^{-t}\cos 5t$ ➔ -2+0

MOST OF these signals have an asymptotic value of ZERO – only the constants remain non-zero!
SUMMARY: The only signal with a non-zero asymptotic value is a constant!

HEAVISIDE STEP FUNCTION – this is a convenient mathematical mechanism for representing both:

1. A step input
2. A constant in future time.

$$H(t)=0 \quad t<0 \qquad H(t)=1 \quad t\ge 0 \qquad L[H(t)]=\frac{1}{s}$$



INTERIM SUMMARY: A convergent signal will have a non-zero asymptotic value (or steady-state) if and only if it contains a step function, that the partial fraction expansion includes a term (A/s). By inspection, the asymptotic value will be A – assuming all other components converge to zero. The steady-state can be determined by doing the partial fraction expansion and finding A.
FINAL VALUE THEOREM (simple shortcut to find A) **WARNING**: FVT can only be applied if the signal is convergent. If applied to the Laplace Transform of a divergent signal, the answer will be meaningless.

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}(sF(s))$	Students may note this is the same as using the cover-up rule to find the residue for the term (A/s) in the partial fraction expansion.

EXAMPLES of applying the FVT showing consistency between time domain and result	
$\left\{x(t)=e^{-at};\quad X(s)=\frac{1}{s+a}\right\}$	$\left\{\lim_{t\to\infty} x(t)=0;\quad \lim_{s\to 0} sX(s)=\frac{0}{a}=0\right\}$
$\left\{y(t)=1-e^{-bt};\quad Y(s)=\frac{b}{s(s+b)}\right\}$	$\left\{\lim_{t\to\infty} y(t)=1;\quad \lim_{s\to 0} sY(s)=\frac{b}{b}=1\right\}$
$\left\{z(t)=e^{-t}\sin 2t;\quad Z(s)=\frac{2}{(s+1)^2+4}\right\}$	$\left\{\lim_{t\to\infty} z(t)=0;\quad \lim_{s\to 0} sZ(s)=\frac{0}{5}=0\right\}$
$\left\{w(t)=te^{at};\quad X(s)=\frac{1}{s(s-a)}\right\}$	$\left\{\lim_{t\to\infty} z(t)=\infty;\quad \lim_{s\to 0} sZ(s)=\frac{0}{-a}=0\right\}$ **Inconsistent because signal not convergent so FVT not valid!**

EXAMPLES of applying the FVT and links to partial fraction expansion Suggest you validate these with MATLAB, e.g. y=impulse(tf([1 6],[1 4 3]),1000);y(end)	
$G(s)=\frac{5}{(s+1)(s^2+s+4)(s+2)}=\frac{A}{s+1}+\frac{2B+Cs}{s^2+s+4}+\frac{D}{s+2}$	$\left\{\lim_{s\to 0} sG(s)=\frac{0}{8}=0\right\}$ No factor (1/s) so expect zero.
$H(s)=\frac{5}{s(s^2+s+4)(s+2)}=\frac{A}{s}+\frac{2B+Cs}{s^2+s+4}+\frac{D}{s+2}$	$\left\{\lim_{s\to 0} sH(s)=\frac{5}{8}=A\right\}$ Same result as cover up rule!
$L(s)=\frac{s+6}{s(s+2)(s^2+3s+7)}$	$\left\{\lim_{s\to 0} sL(s)=\frac{6}{14}\right\}$
$M(s)=\frac{s+6}{(s-4)(s+2)(s^2+3s+7)}$	$\left\{\lim_{s\to 0} sM(s)=\frac{0}{-56}=0\right\}$ INCORRECT because signal not convergent so FVT not valid!
$M(s)=\frac{s+6}{(s+4)(s^2+7)}$	$\left\{\lim_{s\to 0} sN(s)=\frac{0}{26}=0\right\}$ INCORRECT because signal not convergent – it is sinusoidal!

REMINDER: If a transform does not include a single factor ‘s’ in the denominator, the final value will be zero or infinite (except for pure sinusoids). FVT only applies to convergent signals.