Modelling and control summaries

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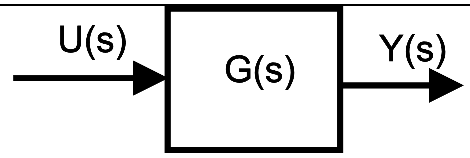
Offset 2 – system gain and closed-loop

This summary assumes that any implied signals are convergent and hence the final value theorem (FVT) can be applied – see offset 1.

DEFINITION OF SYSTEM STEADY-STATE GAIN

Assume that the system input u(t) is constant. The steady-state gain is the ratio of the asymptotic output signal to the input signal assuming the output is convergent. For convenience use u(t)=1, so gain becomes asymptotic output.



The FVT can be used to determine the system gain by assuming that U(s)=(1/s) – a unit step.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{G(s)}{s} = \lim_{s\to 0} G(s) = G(0)$$

REMARK: System gain G(0) is well defined even for unstable G(s) because a convergent u(t) can be chosen to cancel any divergent dynamics and hence give a convergent output.

What is the system gain for the following transfer functions?

$$\left\{H(s) = \frac{A}{s+1} \quad \Rightarrow \quad H(0) = A\right\} \quad \left\{G(s) = \frac{5}{(s+1)(s^2+s+4)(s+2)} \quad \Rightarrow \quad G(0) = \frac{5}{8}\right\}$$

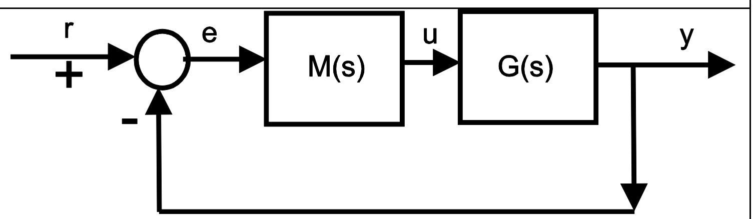
$$\left\{M(s) = \frac{4}{s(s+3)} \quad \Rightarrow \quad M(0) = \infty\right\} \quad \left\{K(s) = \frac{A}{s-1} \quad \Rightarrow \quad K(0) = -A\right\}$$

You can verify these for H, G, M by using step.m in MATLAB and finding asymptotic output.

DEFINITION OF SYSTEM CLOSED-LOOP STEADY-STATE GAIN

First determine the closed-loop transfer functions and decide which gain you want, that is the gain between which signals.

1. Closed-loop gain is typically defined by linking the target r to the output y, that is G_c.
2. Offset is e=(r-y) and more often users are interested in this as it is the steady-state error between target and output, that is G_{ce}.



$$y = \underbrace{\left(\frac{GM}{1+GM}\right)}_{G_c} r; \quad u = \frac{M}{1+GM} r; \quad e = \underbrace{\left(\frac{1}{1+GM}\right)}_{G_{ce}} r$$

REMARK: This note will consider the computation of both system gain and offset. For convenience, offset is normally defined for a unit target, that is r=1. HENCE, as gain is equivalent to asymptotic output signal value, then:

GAIN=$G_c(0)$
OFFSET=$G_{ce}(0)$

SHORTCUT: $G_c(0) = \left(\frac{G(0)M(0)}{1+G(0)M(0)}\right)$; $G_{ce}(0) = \left(\frac{1}{1+G(0)M(0)}\right)$

EXAMPLES - assuming a standard closed-loop configuration, find the steady-state system gain and steady-state offsets (for unit target r=1) for the following. FIRST you must check that the closed-loop is stable (by all means confirm with MATLAB or otherwise).	
$G(s) = \frac{3}{(s-1)(s+5)}$; $M(s) = 2$	$G_c(0) = \left(\frac{6/-5}{1+6/-5}\right) = 6$; $G_{ce}(0) = \left(\frac{1}{1+6/-5}\right) = -5$
$G(s) = \frac{s+6}{(s+4)(s+2)}$; $M(s) = 2$	$G_c(0) = \left(\frac{12/8}{1+12/8}\right) = \frac{3}{5}$; $G_{ce}(0) = \left(\frac{1}{1+12/8}\right) = \frac{2}{5}$
$\left\{G(s) = \frac{1}{(s-4)};\ M(s) = 2\right\} \Rightarrow G_c = \frac{2}{s-4+2}$ What M would give a LHP pole – try this and then find gain/offset?	$G_c(0) = \left(\frac{2/-4}{1+2/-4}\right) = -1$; $G_{ce}(0) = \left(\frac{1}{1+2/-4}\right) = 2$ **INVALID: Closed-loop pole is in RHP so signals not convergent!**
$G(s) = \frac{s+1}{s(s+3)}$; $M(s) = 4$	$G_c(0) = \left(\frac{4/0}{1+4/0}\right) = 1$; $G_{ce}(0) = \left(\frac{1}{1+4/0}\right) = 0$ **Note use of integrator has made the steady-state gain G(0)→∞ and this affects result.**

SUMMARY:

1. System steady-state gain can be computed by substituting s=0 into a transfer function.
2. The steady-state gain of a loop can be computed by substituting s=0 into a closed-loop transfer function.
3. Steady-state output is **gain** x **steady state input**.
4. Signals must be convergent so check for stability.
5. If a transfer function contains an integrator, the steady-state gain is infinite.

CHALLENGE: How would system closed-loop gain and offset calculations change for a feedback loop with sensor dynamics such as the one below?