Modelling and control summaries

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Inverse Laplace – tutorial sheet

1. The inverse Laplace transform of a transfer function is called the "impulse response function". If a system has an impulse response function given by g(t) = e^{-t}(1-sin(t)). Compute its transfer function, G(s).

2. Use Laplace methods to solve the following ODE equations.

$$R\frac{dq}{dt}+\frac{q}{C}=0.4; \quad \frac{dx}{dt}+0.01x=e^{-0.02t}; \quad \frac{dy}{dt}+y=t; \quad 100\frac{dh}{dt}+h=25;$$

3. Give examples of type 0, type 1 and type 2 systems. Has does this affect the expected behaviour?

4. Which of the following transforms for 1st order ODEs has the highest gain? What are the gains? What are the time constants?

$$\frac{1}{s+0.25}; \quad \frac{15}{(s+5)}; \quad \frac{6}{5s+4}; \quad \frac{9}{4s+8};$$

   Determine and sketch the step responses for each of these.

5. An approximation to the equation of motion of the rotational dynamics of a rocket modelled as a rigid body, moving through the atmosphere is given by:

$$J\frac{d^2\theta}{dt^2}-NL\theta=\tau(t)$$

   where θ is the vehicle's angle of attitude, J is its rotational inertia, N is the normal force coefficient, L is the offset of the centre of pressure (CoP) from the centre of gravity (CoG) and τ(t) is the applied torque, e.g. from a thruster. For simplicity, it is assumed there is no friction so no aerodynamic damping. L can be positive or negative depending on whether the CoP is in front of or behind the CoG.
   i. Find the transfer function between τ (s) and θ (s).
   ii. Under what condition is the system unstable?

6. The frequency of signals, b(t), travelling from baroreceptors that sense changes in arterial pressure to the vasomotor centre in the brain is directly proportional to the arterial pressure, p(t) plus the rate of change of that pressure with respect to time.
   i. Set up the differential equation relating b(t) to p(t).
   ii. Write down the transfer function between B(s) and P(s).

7. A simple accelerometer comprises a mass suspended in series with a spring and a damper inside a case. The equation describing the motion of the mass is:

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=m\frac{d^2x}{dt^2}$$

where m is the mass, c is the damping constant and k is the spring constant, y(t) is the displacement of the mass and x(t) is the displacement of the base whose acceleration, is to be measured.

a. Find the transfer function between the base acceleration, A(s), and the mass displacement, Y (s).
b. Find the transfer function between the base acceleration, A(s), and the mass acceleration.

8. The sugar levels in a working muscle working at a constant rate are given by.

$$0.03\frac{dc}{dt}+0.004c=0.05;$$

Use Laplace methods to determine level as a function to time given an initial level of 15.

9. Convert the following into Laplace transforms and hence solve.

$$\frac{dw}{dt}+4w=1;\quad 3\frac{dz}{dt}-2z=e^{-2t};\quad \frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=t$$

$$\frac{dx}{dt}+0.2x=4e^{-0.5t};\quad \frac{dy}{dt}+2y=1+0.4\cos t$$

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=3;\quad \frac{d^2z}{dt^2}+0.2\frac{dz}{dt}+1.01z=4\frac{dx}{dt}+5x;\quad 3\frac{dy}{dt}+6y=z$$

10. Expand into partial fractions:

$$G=\frac{5}{s^2+6s+8};\quad G=\frac{10}{s^2+s-2};\quad G=\frac{s^2+s+5}{(s+2)^2(s+4)};\quad G=\frac{s-2}{(s+4)^3(s-1)};$$

11. Use a table of Laplace transforms to determine the inverse Laplace transforms of the following:

$$F(s)=\frac{2}{s};\quad G(s)=\frac{3s+6}{s^2+4};\quad H(s)=\frac{16}{(s+2)(s+4)};\quad M(s)=\frac{2s-3}{s^2+4s+8}$$

$$N(s)=\frac{s+4}{(s+2)^2 s};\quad K(s)=\frac{4s+5}{(s^2+6s+10)(s+1)}$$

Outline Answers

1. The inverse Laplace transform of a transfer function is called the "impulse response function". If a system has an impulse response function given by g(t) = e^{-t}(1-sin(t)). Compute its transfer function, G(s).

$$L[e^{-t}(1-\sin(t))] = L[e^{-t} - e^{-t}\sin(t))] = L[e^{-t}] - L[e^{-t}\sin(t))]$$

$$= \frac{1}{s+1} - \frac{1}{(s+1)^2+1} = \frac{s^2+s+1}{(s+1)[(s+1)^2+1]}$$

2. Use Laplace methods to solve the following ODE equations.

$$R\frac{dq}{dt} + \frac{q}{C} = 0.4; \quad \frac{dx}{dt} + 0.01x = e^{-0.02t}; \quad \frac{dy}{dt} + y = \cos 2t; \quad 100\frac{dh}{dt} + h = 25;$$

$$Q(s) = \frac{0.4}{s(Rs+\frac{1}{C})} = \frac{0.4/R}{s(s+\frac{1}{CR})} = \frac{0.4}{R}\left(\frac{CR}{s} - \frac{CR}{s+\frac{1}{CR}}\right); \quad q(t) = 0.4C(1-e^{-t/RC})$$

$$X(s) = \frac{1}{(s+0.01)(s+0.02)} = \frac{100}{s+0.01} - \frac{100}{s+0.02}; \quad x(t) = 100(e^{-0.01t} - e^{-0.02t})$$

$$Y(s) = \frac{1}{(s+1)s^2} = \frac{As+B}{s^2} + \frac{C}{s+1}; \quad y(t) = Ce^{-t} + A + Bt$$

$$H(s) = \frac{0.25}{(s+0.01)s} = \frac{25}{s} - \frac{25}{s+0.01}; \quad h(t) = 25(1-e^{-0.01t})$$

3. Bookwork

4. Which of the following transforms for 1st order ODEs has the highest gain? What are the gains? What are the time constants?

$$\frac{1}{s+0.25}; \quad \frac{15}{(s+5)}; \quad \frac{6}{5s+4}; \quad \frac{9}{4s+8};$$

Determine and sketch the step responses for each of these.

Gains are 4, 3, 1.5 and 1.125 respectively. Time constants are 4, 0.2, 1.25, 0.5 respectively.

5. $J\frac{d^2\theta}{dt^2} - NL\theta = \tau(t) \quad \Rightarrow \quad [Js^2 - NL]\Theta(s) = \mathrm{T}(s) \quad \Rightarrow \quad \Theta = \frac{1}{Js^2 - NL}\mathrm{T}(s)$

Unstable if any poles are in the RHP. This is the case if L>0.

6. $b = \alpha \frac{dp}{dt} + \beta p \quad \Rightarrow \quad P(s) = \frac{1}{\alpha s + \beta} B(s)$

Unstable if any poles are in the RHP. This is the case if L>0.

7.

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = m\frac{d^2x}{dt^2} \quad \Rightarrow \quad [Ms^2 + cs + k]Y(s) = ms^2X(s); \quad A(s) = s^2X(s)$$

8.

$$\left.\begin{array}{r} 0.03\frac{dc}{dt} + 0.004c = 0.05 \\ c(0) = 15 \end{array}\right\} \quad \Rightarrow \quad C(s) = \frac{1}{0.03s + 0.004}\frac{0.05}{s} + \frac{15 \times 0.003}{0.03s + 0.004}$$

9. Use MATLAB to test your inverse Laplace by ignoring the initial conditions (set to zero).

$$\frac{dw}{dt} + 4w = 1 \quad \Rightarrow \quad (s+1)W(s) - w(0) = \frac{1}{s}$$

$$3\frac{dz}{dt} - 2z = e^{-2t} \quad \Rightarrow \quad (3s-2)Z(s) - 3z(0) = \frac{1}{s+2}$$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = t \quad \Rightarrow \quad (s^2 + 4s + 4)Y(s) - sy(0) - 4y(0) - \dot{y}(0) = \frac{1}{s^2}$$

$$\frac{dx}{dt} + 0.2x = 4e^{-0.5t} \quad \Rightarrow \quad (s+0.2)X(s) - x(0) = \frac{4}{s+0.5}$$

$$\frac{dy}{dt} + 2y = 1 + 0.4\cos t \quad \Rightarrow \quad (s+2)Y(s) - y(0) = \frac{1}{s} + \frac{0.4s}{s^2+1}$$

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 3 \quad \Rightarrow \quad X(s) = \frac{3}{s(s^2 + 4s + 3)}$$

$$\frac{d^2z}{dt^2} + 0.2\frac{dz}{dt} + 1.01z = 4\frac{dx}{dt} + 5x \quad \Rightarrow \quad Z(s) = \frac{4s+5}{s^2 + 0.2s + 1.01}X(s)$$

$$3\frac{dy}{dt} + 6y = z \quad \Rightarrow \quad Y(s) = \frac{1}{3s+6}Z(s)$$

10. Use MATLAB to test your answers.

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