Modelling and control summaries

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Inverse Laplace 2 – partial fractions

TECHNIQUE FOR INVERSE LAPLACE - USE A LOOK-UP TABLE (see Laplace 4)

1. **rearrange the existing Laplace transform into the forms within the table.**
2. **inverse Laplace is then done by inspection**

A key part of this is the partial fraction expansion so students should be skilled in doing this quickly where possible. Here a basic expansion technique is reviewed.

SUMMARY of PARTIAL FRACTION EXPANSION	**Cumbersome but reliable and useful**
STEP 1: Re-write the Laplace transform in terms of its partial fractions by first factorising the denominator. If $r_i(s)$ is a simple factor of the form (s+a) then residue $C_i(s)$ is known to be a constant.	$F(s)=\frac{Q(s)}{P(s)}=\frac{C_1(s)}{r_1(s)}+\frac{C_2(s)}{r_2(s)}+\ldots+\frac{C_n(s)}{r_n(s)}$ Determine residues by multiplying by entire denominator.
STEP 2: Multiply F(s) by the entire denominator. Hence, noting the cancellations as required:	$Q(s)=C_1(s)[r_2r_3\cdots r_n]+C_2(s)[r_1r_3\cdots r_n]+\cdots+C_n(s)[r_1r_2\cdots r_{n-1}]$
STEP 3: Expand both sides and equate coefficients for each power of s.	**This is best demonstrated through some examples.**

Example 1

$F(s)=\frac{4}{s^2+3s+2}=\frac{4}{(s+1)(s+2)}=\frac{C_1}{s+1}+\frac{C_2}{s+2}$	**STEP 1**: Factorise and write partial fraction expansion.
$4=C_1(s+2)+C_2(s+1)=(2C_1+C_2)+s(C_1+C_2)$	**STEP 2**: Multiply both sides by denominator.
$\left\{\underbrace{0=C_1+C_2;}_{coefficients\ of\ s^1}\quad \underbrace{4=2C_1+C_2}_{coefficients\ of\ s^0}\right\} \Rightarrow \begin{matrix} C_1=4 \\ C_2=-4 \end{matrix}$	**STEP 3:** Equate coefficients of powers of 's' to give simultaneous equations in the unknown residues.
HENCE, from the look-up table: $f(t)=4e^{-t}-4e^{-2t}$	

Example 2

$F(s)=\dfrac{5}{s^2+6s+8}=\dfrac{5}{(s+2)(s+4)}=\dfrac{C_1}{s+2}+\dfrac{C_2}{s+4}$	**STEP 1**: Factorise and write partial fraction expansion.
$5=C_1(s+4)+C_2(s+2)=(4C_1+2C_2)+s(C_1+C_2)$	**STEP 2**: Multiply both sides by denominator.
$\left\{\underbrace{0=C_1+C_2;}_{coefficients\ of\ s^1}\quad \underbrace{5=4C_1+2C_2}_{coefficients\ of\ s^0}\right\} \Rightarrow \begin{matrix} C_1=2.5 \\ C_2=-2.5 \end{matrix}$	**STEP 3:** Equate coefficients of powers of 's' to give simultaneous equations in the unknown residues.
HENCE, from the look-up table: $f(t)=2.5e^{-2t}-2.5e^{-4t}$	

Example 3

$F(s)=\dfrac{10}{s(s^2-s-2)}=\dfrac{4}{s(s+1)(s-2)}=\dfrac{C_1}{s}+\dfrac{C_2}{s+1}+\dfrac{C_3}{s-2}$	**STEP 1**: Factorise and write partial fraction expansion.
$10=C_1(s+1)(s-2)+C_2s(s-2)+C_3s(s+1)$ $10=(-2C_1)+s(-C_1-2C_2+C_3)+s^2(C_1+C_2+C_3)$	**STEP 2**: Multiply both sides by denominator.
$\left\{\underbrace{0=C_1+C_2+C_3;}_{coefficients\ of\ s^2}\quad \underbrace{0=-C_1-2C_2+C_3;}_{coefficients\ of\ s^1}\quad \underbrace{10=-2C_1}_{coefficients\ of\ s^0}\right\}$ $\Rightarrow\quad C_1=-5,\quad C_2=\dfrac{10}{3},\quad C_3=\dfrac{10}{6}$	**STEP 3:** Equate coefficients of powers of 's' to give simultaneous equations in the unknown residues.
HENCE, from the look-up table: $f(t)=-5+\dfrac{10}{3}e^{-t}+\dfrac{10}{3}e^{2t}$	

TUTORIAL QUESTIONS: Prove the following

$$\frac{s+2}{(s+3)(s+4)}=\frac{2}{s+4}-\frac{1}{s+3};\qquad \frac{s^2+s-1}{s^3+0.4s^2+0.03s}=-\frac{100/3}{s}-\frac{121/6}{s+0.3}+\frac{109/2}{s+0.1}$$