Modelling and control summaries

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Inverse Laplace 5 – solving ODEs

TECHNIQUE FOR SOLVING ODE USING LAPLACE

1. **take Laplace of every term in the ODE and form Laplace Transform of state.**
2. **rearrange the existing Laplace transform into the forms within the table.**
3. **inverse Laplace is then done as in sheets 1-4!**

This sheet will demonstrate the technique using a number of examples.

EXAMPLE 1 – assume u(t) is a unit step	
STEP 1: Take Laplace of every term in the ODE and form Laplace Transform of state x(t).	$a\frac{dx}{dt}+bx=ku(t) \quad \Rightarrow \quad L[a\frac{dx}{dt}+bx]=L[ku(t)]$ $\Rightarrow \quad a(sX(s)-x(0))+bX(s)=kU(s) \quad \Rightarrow \quad X(s)=\frac{kU(s)+ax(0)}{as+b}$
STEP 2: Substitute U(s)=1/s and rearrange the existing Laplace transform into the forms within the table.	$X(s)=\frac{k}{as+b}\frac{1}{s}+\frac{ax(0)}{as+b}=\frac{x(0)}{s+b/a}+\frac{k/a}{s(s+b/a)}$ Need partial fractions $\Rightarrow \quad X(s)=\frac{x(0)}{s+b/a}+\frac{k/b}{s}-\frac{k/b}{(s+b/a)}$ After expansion

An RC circuit is modelled here. Determine the charge q(t) given an applied voltage v of 12V.	$R\frac{dq}{dt}+\frac{1}{C}q=v(t) \quad R=0.01, \quad C=2\times10^{-5}$
STEP 1: Take Laplace of every term in the ODE and form Laplace Transform of state q(t).	$L[R\frac{dq}{dt}+\frac{1}{C}q]=L[v(t)] \quad \Rightarrow \quad R(sQX(s)-q(0))+\frac{1}{C}Q(s)=\frac{12}{s}$ $\Rightarrow \quad Q(s)=\frac{Rq(0)}{Rs+1/C}+\frac{1}{Rs+1/C}\frac{12}{s}$ Need partial fractions
STEP 2: Rearrange the existing Laplace transform into the forms within the table (make denominator factors monic) and then do partial fractions as required.	$Q(s)=\frac{q(0)}{s+1/RC}+\frac{1/R}{s+1/RC}\frac{12}{s}=\frac{q(0)}{s+1/RC}+\frac{12C}{s}-\frac{12C}{s+1/RC}$ **Hence** $q(t)=(q(0)-12C)e^{-\frac{t}{RC}}+12C=(q(0)-2.4\times10^{-4})e^{-\frac{10^7 t}{2}}+2.4\times10^{-4}$

A suspension unit is modelled here. Determine the displacement x(t) given an applied force f=100N, x(0)=0, dx/dt(0)=0.	$500\dfrac{d^2x}{dt^2}+2000\dfrac{dx}{dt}+1500x=0.2f$
STEP 1: Take Laplace of every term in the ODE and form Laplace Transform of state x(t).	$500[s^2X(s)+4sX(s)+3X(s)]=0.2F(s)+500s\dot{x}(0)+2000x(0)$ $\Rightarrow X(s)=\dfrac{0.2F(s)+500s\dot{x}(0)+2000x(0)}{500[s^2+4s+3]}$ $\Rightarrow X(s)=\dfrac{0.04}{s[s^2+4s+3]}+\dfrac{s\dot{x}(0)+4x(0)}{[s^2+4s+3]}$
STEP 2: Substitute F(s)=100/s, x(0)=0, dx/dt(0) =0, and rearrange the existing Laplace transform into the forms within the table.	$\dfrac{0.04}{s[s^2+4s+3]}=\dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{s+3};\quad \dfrac{s+8}{[s^2+4s+3]}=\dfrac{D}{s+1}+\dfrac{E}{s+3}$ Solve for A,B,C using cover-up rule or otherwise. D,E depend only on initial conditions $x(t)=(A+B+D)e^{-t}+(C+E)e^{-3t}$ $A=\dfrac{1}{75},\ B=\dfrac{1}{50},\quad C=\dfrac{1}{150};\quad D=3.5,\quad E=-2.5$

An tank system with inflow F(t) and depth h is modelled. Determine the depth for a sinusoidal in flow f(t). Assume zero initial conditions.	$4\dfrac{dh}{dt}+0.02h=0.04F;\quad F=0.2\sin 0.01t$
STEP 1: Take Laplace of every term in the ODE and form Laplace Transform of state h(t).	$[4sH(s)+0.02H(s)]=0.04F(s);\quad F(s)=\dfrac{0.01\times 0.2}{s^2+0.01^2}$ $\Rightarrow H(s)=\dfrac{2\times 10^{-5}}{(s+0.005)(s^2+0.01^2)}$
STEP 2: Rearrange the existing Laplace transform into the forms within the table and hence solve. Solve for A,B,C using cover-up rule and expansion.	$H(s)=\dfrac{2\times 10^{-5}}{(s+0.005)(s^2+0.01^2)}=\dfrac{A}{(s+0.005)}+\dfrac{0.01B}{s^2+0.01^2}+\dfrac{Cs}{s^2+0.01^2}$ $A=2\times 10^{-5}/[0.005^2+0.01^2]=2\times 10^{-5}/125\times 10^{-6}=0.16$ $A(s^2+0.01^2)+(0.01B+Cs)(s+0.005)=2\times 10^{-5}$ $C=-A,\quad 0.005C+0.01B=0\quad\Rightarrow\quad B=0.08$ $h(t)=Ae^{-0.005t}+B\sin 0.01t+C\cos 0.01t$