Modelling and control summaries

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Laplace 2 – sinusoids & short cuts

WORKING DEFINITION OF LAPLACE TRANSFORMS F(s) is the Laplace transform of signal *f(t)*	$L[f(t)] \equiv F(s) = \int_0^\infty f(t) e^{-st} dt$

LAPLACE OF SINSUOIDS It is possible to integrate by parts, but this is tedious. Hence, an easier technique is to use the exponential forms for sine and cosine alongside the known result from Laplace 1.	$L[e^{-at}] = \dfrac{1}{s+a}$

$$\sin wt = \frac{e^{jwt} - e^{-jwt}}{2j} \quad \Rightarrow \quad L[\sin wt] = L[\frac{e^{jwt} - e^{-jwt}}{2j}] = \frac{1}{2j}\left[\frac{1}{s-jw} - \frac{1}{s+jw}\right]$$

$$L[\sin wt] = \frac{1}{2j}\left[\frac{(s+jw)-(s-jw)}{(s-jw)(s+jw)}\right] = \frac{1}{2j}\left[\frac{2jw}{(s^2+w^2)}\right] = \frac{w}{s^2+w^2}$$

Key result

$$\cos wt = \frac{e^{jwt} + e^{-jwt}}{2} \quad \Rightarrow \quad L[\cos wt] = L[\frac{e^{jwt} + e^{-jwt}}{2}] = \frac{1}{2}\left[\frac{1}{s-jw} + \frac{1}{s+jw}\right]$$

$$L[\cos wt] = \frac{1}{2}\left[\frac{(s+jw)+(s-jw)}{(s-jw)(s+jw)}\right] = \frac{1}{2}\left[\frac{2s}{(s^2+w^2)}\right] = \frac{s}{s^2+w^2}$$

Key result

REMARK: **The only difference between these two is the numerator but this is critically important so students must remember it**.

TUTORIAL QUESTIONS Convince yourself of the following results.	$L[3\sin 2t] = \dfrac{6}{s^2+4}$; $L[-4\cos 0.2t] = \dfrac{-4s}{s^2+0.04}$; $L[0.2\sin 0.4t] = \dfrac{0.08}{s^2+0.16}$; $L[2-\sin(0.5t)] = \dfrac{2s^2-0.5s+0.5}{s(s^2+0.25)}$; $L[e^t + 2\cos 5t] = \dfrac{3s^2-2s+25}{s^3-s^2+25s-25}$

SHORTCUTS

Using the original integral definition can be tedious for many functions which are products of other functions. Therefore it is best to identify short-cuts.

1. What is the link between *L[f(t)]* and *L[e^{at} f(t)]* ?
2. What is the link between *L[f(t)]* and *L[t f(t)]* ?

MULTIPLY BY AN EXPONENTIAL This shortcut falls out directly from an analysis of the original integral $L[f(t)] = F(s)$ $L[e^{-at} f(t)] = F(s+a)$	$L[f(t)] = \int_0^\infty f(t) e^{-st} dt = F(s)$ $L[e^{-at} f(t)] = \int_0^\infty e^{-at} f(t) e^{-st} dt = \int_0^\infty e^{-(s+a)t} f(t)\, dt$ $\int_0^\infty e^{-(s+a)t} f(t)\, dt = F(s+a)$
TUTORIAL Use the shortcut above to demonstrate the following Laplace transforms are correct.	$L[3te^{-2t}] = \frac{3}{(s+2)^2}; \quad note \quad L[3t] = \frac{3}{s^2};$ $L[e^{-0.2t}\cos(0.5t)] = \frac{s+0.2}{(s+0.2)^2+0.25}; \quad note \quad L[\cos 0.5t] = \frac{s}{s^2+0.25}$ $L[t^2 e^{0.3t}] = \frac{2}{(s-0.3)^3}; \qquad L[3+5e^{-t}\sin 2t] = \frac{3}{s} + \frac{10}{(s+1)^2+4} = \frac{3s^2+16s+15}{s^3+2s^2+5s}$

MULTIPLY BY t This will come up far less often and is given mainly for completeness and also without proof which is available in most engineering mathematics textbooks.	$L[f(t)] = F(s)$ $L[tf(t)] = -\frac{d}{ds} F(s)$
TUTORIAL Use the shortcut above to demonstrate the following Laplace transforms are correct.	$L[t\sin 4t] = -\frac{d}{ds}\left[\frac{4}{s^2+16}\right] = \frac{8s}{(s^2+16)^2}$ $L[t\cos(0.5t)] = -\frac{d}{ds}\left[\frac{s}{s^2+0.25}\right] = \frac{s^2-0.25}{(s^2+0.25)^2}$

SUMMARY: The most important results you will use are the following.

1.

$$L[Ae^{-at}] = \frac{A}{s+a}; \quad L[Ae^{bt}] = \frac{A}{s-b}; \quad L[A] = \frac{A}{s}; \quad L[At^n] = \frac{An!}{s^{n+1}}$$

$$L[A\sin wt] = \frac{Aw}{s^2+w^2}; \quad L[A\cos wt] = \frac{As}{s^2+w^2};$$

$$L[Ae^{-at}\sin wt] = \frac{Aw}{(s+a)^2+w^2}; \quad L[Ae^{-at}\cos wt] = \frac{A(s+a)}{(s+a)^2+w^2};$$

2. Shortcuts can be useful for identifying expected signals directly from a Laplace transform.

- The existence of a factor (s+a) throughout indicates there is an e^{-at} in the signal.
- The existence of a double root in the denominator indicates the existence of a ‘t’ in the signal.