Modelling and control summaries

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Laplace 3 – differential equations

WORKING DEFINITION OF LAPLACE TRANSFORMS F(s) is the Laplace transform of signal *f(t)*	$L[f(t)] \equiv F(s) = \int_0^\infty f(t) e^{-st} dt$

LAPLACE OF DERIVATIVE This is derived by using the integral definition directly and integration by parts.	$\int u \frac{dv}{dt}.dt = uv] - \int v \frac{du}{dt}.dt$

$$L\left[\frac{dx}{dt}\right] = \int_0^\infty \frac{dx}{dt} e^{-st} dt \quad \Rightarrow \quad u = e^{-st}; \quad \frac{du}{dt} = -se^{-st}; \quad \frac{dv}{dt} = \frac{dx}{dt}; \quad v = x$$

$$\Rightarrow \int_0^\infty \frac{dx}{dt} e^{-st} dt = [xe^{-st}]_0^\infty - \int_0^\infty -sxe^{-st} dt = x(0) + sX(s)$$

Key result

REMARK: **The Laplace of derviative is often used to help solve differential equations, that is where one wants an expression for the system state which represents a solution.**

EXAMPLE: Find the Laplace transform of w(t) given it obeys the following ODE.	$3\frac{dw}{dt} + 6w = 9; \quad w(0) = -2$

Take Laplace of every term in the ODE and then rearrange to solve for W(s).

$$\left.\begin{aligned} L\left[\frac{dw}{dt}\right] &= sW(s) - w(0) \\ L[9] &= \frac{9}{s} \end{aligned}\right\} \Rightarrow \quad 3[sW(s) - w(0)] + 6W(s) = \frac{9}{s} \quad \Rightarrow \quad W(s) = \frac{9}{s(3s+6)} + \frac{3w(0)}{3s+6}$$

TUTORIAL

Validate the following transforms for the given ODEs.

$$\left\{2\frac{dw}{dt} - 5w = 4; \quad w(0) = 1\right\} \quad \Rightarrow \quad W(s) = \frac{4}{s(2s-5)} + \frac{2}{2s-5}$$

$$\left\{0.4\frac{dz}{dt} + 0.1z = 0.5u; \quad z(0) = -1\right\} \quad \Rightarrow \quad Z(s) = \frac{0.5}{(0.4s+0.1)}U(s) + \frac{-0.4}{0.4s+0.1}$$

$$\left\{0.2\frac{dp}{dt} - 0.5p = 4e^{-2t}; \quad p(0) = 4\right\} \quad \Rightarrow \quad P(s) = \frac{4}{(0.2s-0.5)(s+2)} + \frac{0.8}{0.2s-0.5}$$

2nd and 3rd order differentials

The result is given here without derivation. Students will find the derivation is straightforward, integrating by parts twice.

$$L\left[\frac{d^2w}{dt^2}\right] = s^2W(s) - sw(0) - \frac{dw}{dt}(0);$$

$$L\left[\frac{d^3w}{dt^3}\right] = s^3W(s) - s^2w(0) - s\frac{dw}{dt}(0) - \frac{d^2w}{dt^2}(0)$$

EXAMPLE: Use Laplace to represent the solution of the following ODE for arbitrary input signal u(t). $e\frac{d^2w}{dt^2}+a\frac{dw}{dt}+bw=ku$	Taking Laplace of every term in the equation gives: $L\left[\frac{d^2w}{dt^2}\right]=(s^2W(s)-sw(0)-\frac{dw}{dt}(0));\quad L\left[\frac{dw}{dt}\right]=sW(s)-w(0)$ $e(s^2W(s)-sw(0)-\frac{dw}{dt}(0))+a(sW(s)-w(0))+bW(s)=kU(s)$ $W(s)=\frac{k}{es^2+as+b}U(s)+\frac{e(sw(0)+\frac{dw}{dt}(0))+aw(0)}{es^2+as+b}$
TUTORIAL Verify the solution for the following ODE with initial value y(0)=2 and initial gradient of 3. $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=e^{-6t}$	$Y(s)=\frac{1}{(s^2+4s+4)(s+6)}+\frac{2s+11}{s^2+4s+4}$

SUMMARY: The most important results you will use are the following.

1.

$$L[x(t)]=X(s);$$
$$L[\frac{dx}{dt}]=sX(s)-x(0)$$
$$L[\frac{d^2x}{dt^2}]=s^2X(s)-sx(0)-\frac{dx}{dt}(0)$$
$$L[\frac{d^3x}{dt^3}]=s^3X(s)-s^2x(0)-s\frac{dx}{dt}(0)-\frac{d^2x}{dt^2}(0)$$

2. In the context of control, transfer functions and feedback, it is common to ignore any initial conditions.