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Differentiation 10
chain rule explanations

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http://controleducation.group.shef.ac.uk/mathematics.html

Introduction

- The first 5 videos introduced differentiation from first principles and product and quotient rules.
- The next very commonly used rule is the chain rule (sometimes function of a function rule).
- This video explains the origins of the rule so students can understand it better.

Students can skip these explanations and go straight to resources 11 if they want to get straight into using the rule.

What is a function of a function

Consider the following two simple functions.

$$y = f(x) = \sin(x); \quad z = g(x) = x^2$$

We can create two different functions from these as follows:

$$h(x) = f(g(x)) = \sin(g(x)) = \sin(x^2);$$
$$k(x) = g(f(x)) = [\sin(x)]^2 = \sin^2 x$$

The terminology 'function of a function' should be clear from this.

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Objective

Are there simple methods for obtaining the derivative of a *function of a function*, or indeed, *function of a function of a function* … .

We will begin from the basic definition of differentiation given earlier.

$$\boxed{\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}}$$

Applying definition to function of a function

Consider a simple function of a function.

$$y = f(x) = \sin(x);$$
$$z = g(x) = x^2$$

$$h(x) = f(g(x)) = \sin(x^2);$$

Now apply the definition.

$$\frac{dh}{dx} = \lim_{\delta x \to 0} \frac{h(x+\delta x) - h(x)}{(x+\delta x) - x}$$

$$\frac{dh}{dx} = \lim_{\delta x \to 0} \frac{f(g(x+\delta x)) - f(g(x))}{\delta x}$$

Applying definition to function of a function

Consider the simple approximation of g(x+δx).

$$\lim_{\delta x \to 0}\left\{ g(x+\delta x) = g(x) + \frac{dg}{dx}\delta x \right\}$$

$$\left\{ \delta g = \frac{dg}{dx}\delta x \right\}$$

Substitute into the derivative of the previous page.

$$\frac{dh}{dx} = \lim_{\delta x \to 0} \frac{f(g(x+\delta x)) - f(g(x))}{\delta x}$$

$$\frac{dh}{dx} = \lim_{\delta x \to 0} \frac{f(g(x)+\delta g) - f(g(x))}{\delta x}$$

Applying definition to function of a function

Next, consider the derivative of the function f(x) only.

$$\lim_{\delta g \to 0} \left\{ f(g+\delta g) = f(g) + \frac{df}{dg}\delta g \right\}$$

Use g as this is argument in function f.

Substitute into the derivative of the previous page.

$$\frac{dh}{dx} = \lim_{\delta x \to 0} \frac{f(g+\delta g) - f(g)}{\delta x}$$

$$\frac{dh}{dx} = \lim_{\delta x \to 0} \frac{f(g) + \frac{df}{dg}\delta g - f(g)}{\delta x} = \lim_{\delta x \to 0} \frac{\frac{df}{dg}\delta g}{\delta x}$$

Applying definition to function of a function

Recall definition of δg from 2 pages ago:

$$\frac{dh}{dx}=\lim_{\delta x\to 0}\frac{\frac{df}{dg}\delta g}{\delta x}; \qquad \delta g=\frac{dg}{dx}\delta x$$

AND HENCE WITH

$$h(x)=f(g(x))$$

$$\frac{dh}{dx}=\lim_{\delta x\to 0}\frac{\frac{df}{dg}\frac{dg}{dx}\delta x}{\delta x}=\frac{df}{dg}\frac{dg}{dx}$$

This is called the chain rule.

Differentiation with multiple functions

What if functions are nested multiple times over. It is straightforward to show.

$$h(x) = f(g(k(x))) \quad \Rightarrow \quad \frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dk}\frac{dk}{dx}$$

$$h(x) = f(g(k(w(x)))) \quad \Rightarrow \quad \frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dk}\frac{dk}{dw}\frac{dw}{dx}$$

This is called the chain rule as each function is differentiated in order (like a chain), where the order is linked to the nesting.

Example 1

Find the derivative of h(x).

$$y = f(x) = \sin(x);$$

$$z = g(x) = x^2$$

$$h(x) = f(g(x)) = \sin(x^2);$$

From the chain rule:

$$\frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$$f = \sin(g) \quad \Rightarrow \quad \frac{df}{dg} = \cos(g)$$

$$g = x^2 \quad \Rightarrow \quad \frac{dg}{dx} = 2x$$

$$\frac{dh}{dx} = \cos(g)2x = 2x\cos(x^2)$$

Example 2

Find the derivative of h(x).

$$f(g) = \tan(2g); \quad g(w) = \log(3w); \quad w(x) = e^{2x} + 1$$

$$h(x) = f(g(w(x))) = \tan(2\underbrace{\log(3\underbrace{(e^{2x}+1)}_{w})}_{g});$$

From the chain rule:

$$f = \tan(2g) \quad \Rightarrow \quad \frac{df}{dg} = 2\sec^2(2g)$$

$$g = \log(3w) \quad \Rightarrow \quad \frac{dg}{dw} = \frac{1}{w}$$

$$w = e^{2x} + 1 \quad \Rightarrow \quad \frac{dw}{dx} = 2e^{2x}$$

$$\frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dw}\frac{dw}{dx}$$

$$\frac{dh}{dx} = 2\sec^2(2g).\frac{1}{w}.2e^{2x}$$

Appendix

An easy short hand to remember for simple functions of functions is given below **[u=u(x)].**

$$h=\sin(u) \quad\Rightarrow\quad \frac{dh}{dx}=\frac{dh}{du}\frac{du}{dx}=\cos(u)\frac{du}{dx}$$

$$h=\cosh(u) \quad\Rightarrow\quad \frac{dh}{dx}=\frac{dh}{du}\frac{du}{dx}=\sinh(u)\frac{du}{dx}$$

$$h=\log(u) \quad\Rightarrow\quad \frac{dh}{dx}=\frac{dh}{du}\frac{du}{dx}=\frac{1}{u}\frac{du}{dx}$$

Summary

- This brief resource has derived the chain rule for differentiating functions that are nested expressions of other functions.
- For example:

$$h(x) = f(g(k(w(x)))) \quad \Rightarrow \quad \frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dk}\frac{dk}{dw}\frac{dw}{dx}$$

- The next video will give numerous examples.

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