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Differentiation 11
chain rule worked examples

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http://controleducation.group.shef.ac.uk/mathematics.html

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Introduction

- The previous video gave an explanation of and definition for the chain rule.
- This video will give several worked examples demonstrating the use of the chain rule (sometimes function of a function rule).

$$h(x) = f(g(x)) \quad \Rightarrow \quad \frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$$h(x) = f(g(k(x))) \quad \Rightarrow \quad \frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dk}\frac{dk}{dx}$$

Example 1

Find the derivative of h(x).

$$f(x) = \sec(x); \quad z = g(x) = 4x^2$$

$$h(x) = f(g(x)) = \sec(4x^2)$$

From the chain rule:

$$\frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$$f = \sec(g) \quad \Rightarrow \quad \frac{df}{dg} = \frac{\sin(g)}{\cos^2(g)}$$

$$g = 4x^2 \quad \Rightarrow \quad \frac{dg}{dx} = 8x$$

$$\frac{dh}{dx} = \frac{\sin(g)}{\cos^2(g)} 8x = 8x \frac{\sin(4x^2)}{\cos^2(4x^2)}$$

Example 2 Find the derivative of h(x).

$$f(g)=\cot(2g); \quad g(w)=w^3+3w; \quad w(x)=\log(4x-2)$$

$$h(x)=f(g(w(x)))=\cot(2\underbrace{([\log(4x-2)]^3+3[\log(4x-2)])}_{g});$$

From the chain rule:

$$\frac{dh}{dx}=\frac{df}{dg}\frac{dg}{dw}\frac{dw}{dx}$$

$$f=\cot(2g) \quad \Rightarrow \quad \frac{df}{dg}=-2\operatorname{cosec}^2(2g)$$

$$g=w^3+3w \quad \Rightarrow \quad \frac{dg}{dw}=3w^2+3$$

$$w=\log(4x-2) \Rightarrow \quad \frac{dw}{dx}=\frac{4}{4x-2}$$

$$\frac{dh}{dx}=-2\operatorname{cosec}^2(2g).(3w^2+3).\frac{4}{4x-2}$$

Example 3

Find the derivative of f(x).

Here the user needs to discern that there are nested functions.

$$f(x) = \cos(2x^3)$$

$$f = \cos(g); \quad g = 2x^3$$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$$\frac{df}{dx} = -\sin(g).6x^2 = -\sin(2x^3)6x^2$$

Example 4

Find the derivative of f(x).

Here the user needs to discern that there are nested functions.

$$f(x) = 2\tan(e^{4x^4 - x^2})$$

$$f = 2\tan(g); \quad g = e^v; \quad v = 4x^4 - x^2$$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dv}\frac{dv}{dx}$$

$$\frac{df}{dx} = 2\sec^2(g).e^v(16x^3 - 2x)$$

$$\frac{df}{dx} = 2\sec^2(e^{4x^4 - x^2}).e^{4x^4 - x^2}.(16x^3 - 2x)$$

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Example 5

Find the derivative of f(x).

$$f(x) = 2\sin\left(\frac{e^x}{x+2}\right)$$

Here the user needs to discern that there are nested functions and the quotient rule is also needed!.

$$f = 2\sin(g); \quad g = \frac{e^x}{x+2} = \frac{u}{v}$$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$$\frac{dg}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dg}{dx} = \frac{(x+2)e^x - e^x}{(x+2)^2} = \frac{(x+1)e^x}{(x+2)^2}$$

$$\frac{df}{dx} = 2\cos\left(\frac{e^x}{x+2}\right) \cdot \left(\frac{(x+1)e^x}{(x+2)^2}\right)$$

Summary

This brief resource has given a number of worked examples of using the chain rule.

$$h(x) = f(g(k(w(x)))) \quad \Rightarrow \quad \frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dk}\frac{dk}{dw}\frac{dw}{dx}$$

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