



Differentiation 11

chain rule worked examples

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<http://controleducation.group.shef.ac.uk/mathematics.html>

Introduction

- The previous video gave an explanation of and definition for the chain rule.
- This video will give several worked examples demonstrating the use of the chain rule (sometimes function of a function rule).

$$h(x) = f(g(x)) \quad \Rightarrow \quad \frac{dh}{dx} = \frac{df}{dg} \frac{dg}{dx}$$

$$h(x) = f(g(k(x))) \quad \Rightarrow \quad \frac{dh}{dx} = \frac{df}{dg} \frac{dg}{dk} \frac{dk}{dx}$$

Example 1

Find the derivative of $h(x)$.

$$f(x) = \sec(x); \quad z = g(x) = 4x^2$$

$$h(x) = f(g(x)) = \sec(4x^2)$$

From the chain
rule:

$$\frac{dh}{dx} = \frac{df}{dg} \frac{dg}{dx}$$

$$f = \sec(g) \Rightarrow \frac{df}{dg} = \frac{\sin(g)}{\cos^2(g)}$$

$$g = 4x^2 \Rightarrow \frac{dg}{dx} = 8x$$

$$\frac{dh}{dx} = \frac{\sin(g)}{\cos^2(g)} 8x = 8x \frac{\sin(4x^2)}{\cos^2(4x^2)}$$

Example 2

Find the derivative of $h(x)$.

$$f(g) = \cot(2g); \quad g(w) = w^3 + 3w; \quad w(x) = \log(4x - 2)$$

$$h(x) = f(g(w(x))) = \cot(2(\underbrace{[\log(4x - 2)]^3 + 3[\log(4x - 2)]}_g));$$

From the chain rule:

$$\frac{dh}{dx} = \frac{df}{dg} \frac{dg}{dw} \frac{dw}{dx}$$

$$f = \cot(2g) \Rightarrow \frac{df}{dg} = -2 \operatorname{cosec}^2(2g)$$

$$g = w^3 + 3w \Rightarrow \frac{dg}{dw} = 3w^2 + 3$$

$$w = \log(4x - 2) \Rightarrow \frac{dw}{dx} = \frac{4}{4x - 2}$$

$$\frac{dh}{dx} = -2 \operatorname{cosec}^2(2g) \cdot (3w^2 + 3) \cdot \frac{4}{4x - 2}$$

Example 3

Find the derivative of $f(x)$.

$$f(x) = \cos(2x^3)$$

$$f = \cos(g); \quad g = 2x^3$$

Here the user needs to discern that there are nested functions.

$$\frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx}$$

$$\frac{df}{dx} = -\sin(g) \cdot 6x^2 = -\sin(2x^3) 6x^2$$

Example 4

Find the derivative of $f(x)$.

$$f(x) = 2 \tan(e^{4x^4 - x^2})$$

Here the user needs to discern that there are nested functions.

$$f = 2 \tan(g); \quad g = e^v; \quad v = 4x^4 - x^2$$

$$\frac{df}{dx} = \frac{df}{dg} \frac{dg}{dv} \frac{dv}{dx}$$

$$\frac{df}{dx} = 2 \sec^2(g) \cdot e^v (16x^3 - 2x)$$

$$\frac{df}{dx} = 2 \sec^2(e^{4x^4 - x^2}) \cdot e^{4x^4 - x^2} \cdot (16x^3 - 2x)$$

Example 5

Find the derivative of $f(x)$.

$$f(x) = 2 \sin\left(\frac{e^x}{x+2}\right)$$

Here the user needs to discern that there are nested functions and the quotient rule is also needed!

$$f = 2 \sin(g); \quad g = \frac{e^x}{x+2} = \frac{u}{v}$$

$$\frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx}$$

$$\frac{dg}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{dg}{dx} = \frac{(x+2)e^x - e^x}{(x+2)^2} = \frac{(x+1)e^x}{(x+2)^2}$$

$$\frac{df}{dx} = 2 \cos\left(\frac{e^x}{x+2}\right) \cdot \left(\frac{(x+1)e^x}{(x+2)^2}\right)$$

Summary

This brief resource has given a number of worked examples of using the chain rule.

$$h(x) = f(g(k(w(x)))) \Rightarrow \frac{dh}{dx} = \frac{df}{dg} \frac{dg}{dk} \frac{dk}{dw} \frac{dw}{dx}$$



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