The University Of Sheffield.

Differentiation 12
implicit differentation explanations

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Introduction

- So far these resources have focussed on cases where the function is simply defined as y=f(x), w=g(z) or similar.
- That is a single explicit output and a single independent variable.
- Sometimes the relationship between 2 variables is not explicit and cannot be solved algebraically, for example:

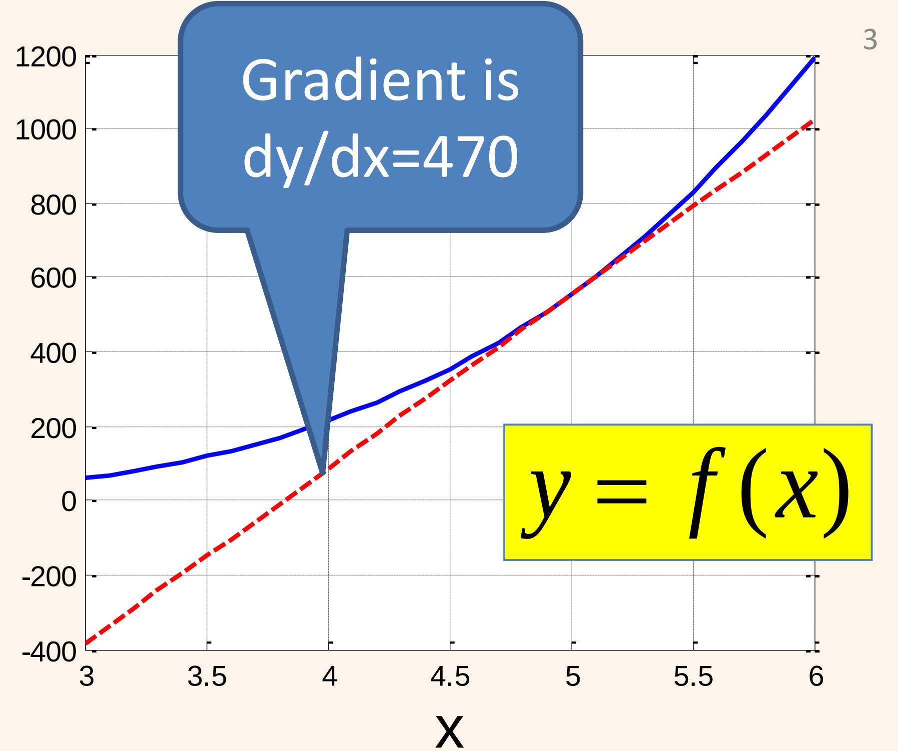
$$y + \sin(y) = x^2 - \cos(x+1)$$

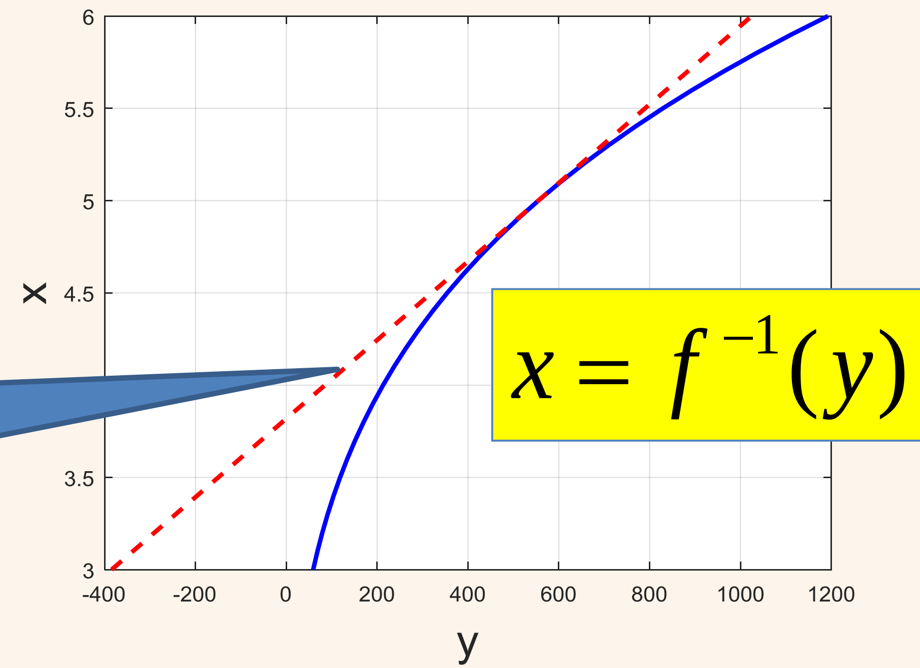
How do we find the derivative of such a curve?

Observation

dy/dx is the gradient of the curve with x on the horizontal axis and y on the vertical axis.

A simple swapping of the axis reveals that the gradient is of course the inverse.





Assumption

Consider a function written as: $y = f(x)$

Assume the inverse function exists, so that we can easily differentiate to find dx/dy: $x = f^{-1}(y)$

It is obvious that the gradients of each of these curves are the inverses one of another, as in essence they are the same curve but with swapped axis.

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad or \quad \frac{dy}{dx} . \frac{dx}{dy} = 1$$

In fact, students can guess that this must be true, even if a simple inverse function does not exist.

Simple logarithm function

Find the derivative of the function $y = \log(x)$

STEP 1: Write x as a function of y (the inverse function). $x = e^y$

STEP 2: Differentiate wrt to y. $\frac{dx}{dy} = e^y$

STEP 3: Use inverse to find dy/dx.

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}$$

Inverse sine

Find the gradient dy/dx of a curve defined with the following expression.

$$y = \sin^{-1} ax$$

METHOD: First use the inverse function.

$$\frac{\sin(y)}{a} = x$$

Now differentiate wrt x.

$$\frac{\cos(y)}{a} = \frac{dx}{dy}$$

$$\frac{dy}{dx} = \frac{a}{\cos(y)} = \frac{a}{\sqrt{1-\sin^2 y}} = \frac{a}{\sqrt{1-(ax)^2}}$$

Using the chain rule on y

Let the expression contain functions of the ‘dependent variable’ y.

Using the chain rule the following is obvious.

$$\frac{d}{dx}(f(y)) = \frac{df}{dy}.\frac{dy}{dx}$$

Easy examples:

$$\frac{d}{dx}(y^2) = 2y.\frac{dy}{dx}$$

$$\frac{d}{dx}(\sin(y)) = \cos(y).\frac{dy}{dx}$$

Implicit differentiation

We will make use of two observations:

- The fact that (dy/dx) . (dx/dy) =1
- The chain rule.

Using these we can differentiate an expression containing two variables and separate out a derivative.

Example 1

Find the gradient of a curve defined with the following expression.

$$y + \cos(2y) = x^3 + x$$

NOTE: The answer includes values from both x and y!

METHOD: Differentiate every term in the expression, using the chain rule for any terms which include a 'y'.

$$\frac{dy}{dx} - 2\sin(2y)\frac{dy}{dx} = 3x^2 + 1$$

Separate variables:

$$\frac{dy}{dx} = \frac{3x^2 + 1}{1 - 2\sin(2y)}$$

Example 2 – a circle

Find the gradient of a curve defined with the following expression.

$$y^2 + x^2 = 4$$

METHOD: Differentiate every term in the expression, using the chain rule for any terms which include a ‘y’.

$$2y\frac{dy}{dx} + 2x = 0$$

Separate variables:

$$\frac{dy}{dx} = \frac{-x}{y}$$

NOTE: The answer includes values from both x and y!

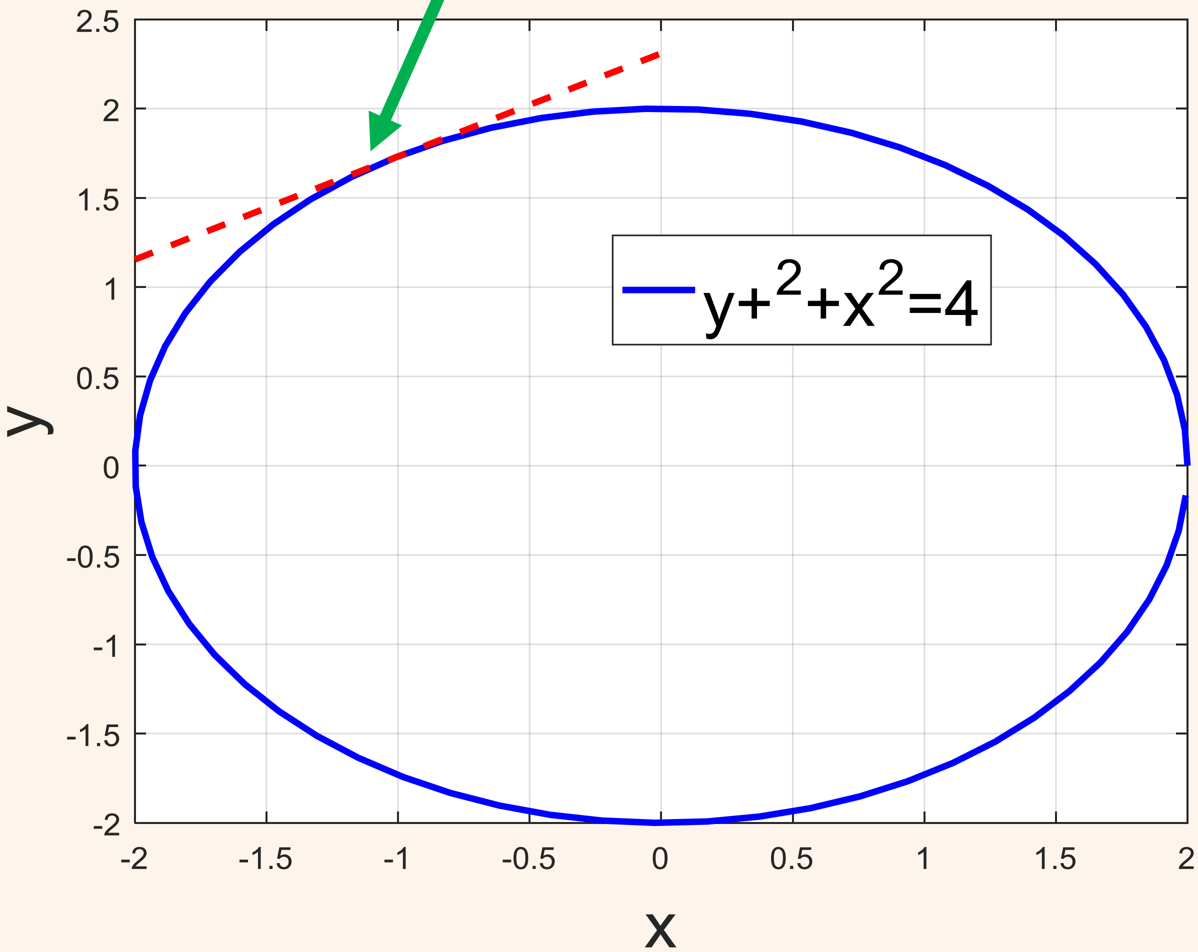
Circle

It is easy to show that this formulae works.

$$\begin{Bmatrix} x=-1 \\ y=\sqrt{3} \end{Bmatrix} \Rightarrow \frac{dy}{dx}=\frac{-1}{\sqrt{3}}$$

$$y^2+x^2=4$$

$$\frac{dy}{dx}=\frac{-x}{y}$$



Example 2 – observation

In this case one could write down an expression for y as follows.

$$y^2 + x^2 = 4 \quad \Rightarrow \quad y = \sqrt{x^2 - 4}$$

Hence one can deduce:

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 4}}$$

Same answer as on the previous page:

$$\frac{dy}{dx} = \frac{-x}{y}$$

Example 3 – back to front

Find the gradient dy/dx of a curve defined with the following expression.

$$y^2 + y^4 + \cos(y) = 4x$$

METHOD: Differentiate every term in the expression, using the chain rule for any terms which include a 'y'.

$$[2y + 4y^3 - \sin(y)]\frac{dy}{dx} = 4$$

Separate variables:

$$\frac{dy}{dx} = \frac{4}{[2y + 4y^3 - \sin(y)]}$$

Example 3 – alternative

Find the gradient dy/dx of a curve defined with the following expression.

$$y^2 + y^4 + \cos(y) = 4x$$

METHOD: Differentiate with respect to y and then note the relationship between dy/dx and dx/dy.

$$[2y + 4y^3 - \sin(y)] = 4\frac{dx}{dy}$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{4}{[2y + 4y^3 - \sin(y)]}$$

Example 4 – logarithms

Find the gradient dy/dx of a curve defined with the following expression.

$$y = \log(x^2 + 3x - 1)$$

METHOD: First express using exponentials.

$$e^y = x^2 + 3x - 1$$

Now differentiate every term.

$$e^y \frac{dy}{dx} = 2x + 3$$

$$\frac{dy}{dx} = \frac{2x+3}{e^y} = \frac{2x+3}{x^2+3x-1}$$

Observation

Find the gradient dy/dx of a curve defined with the following expression.

$$y = \log(f(x))$$

METHOD: First express using exponentials.

$$e^y = f(x)$$

Now differentiate every term.

$$e^y \frac{dy}{dx} = \frac{df}{dx}$$

$$\frac{dy}{dx} = \frac{{df}/{dx}}{e^y} = \frac{{df}/{dx}}{f(x)}$$

Summary

- This brief resource has derived the rule for implicit differentiation.
- This is needed when the output or dependent variable is not defined explicitly in terms of the independent variable and is also **useful for inverse functions**.
- The basic method is to differentiate every term in the expression, using the chain rule as required, and then separate out the terms which include the derivative.
- At times recognising (dy/dx). (dx/dy) =1 can be useful.

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