



Differentiation 13

parametric equations

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Introduction

- So far these resources have focussed on cases where the function is simply defined as $y=f(x)$, $w=g(z)$ or similar.
- That is a single expression with two variables.
- Sometimes the relationship between 2 variables is expressed through parametric equations, that is involving a 3rd variable, e.g.:

$$y = \sin(\omega t); \quad x = \cos(\omega t)$$

How do we find the derivative of such a curve?

Parametric equations

These are quite common, for example students will be familiar with the equation of a circle.

$$x = \cos(t) - 1$$

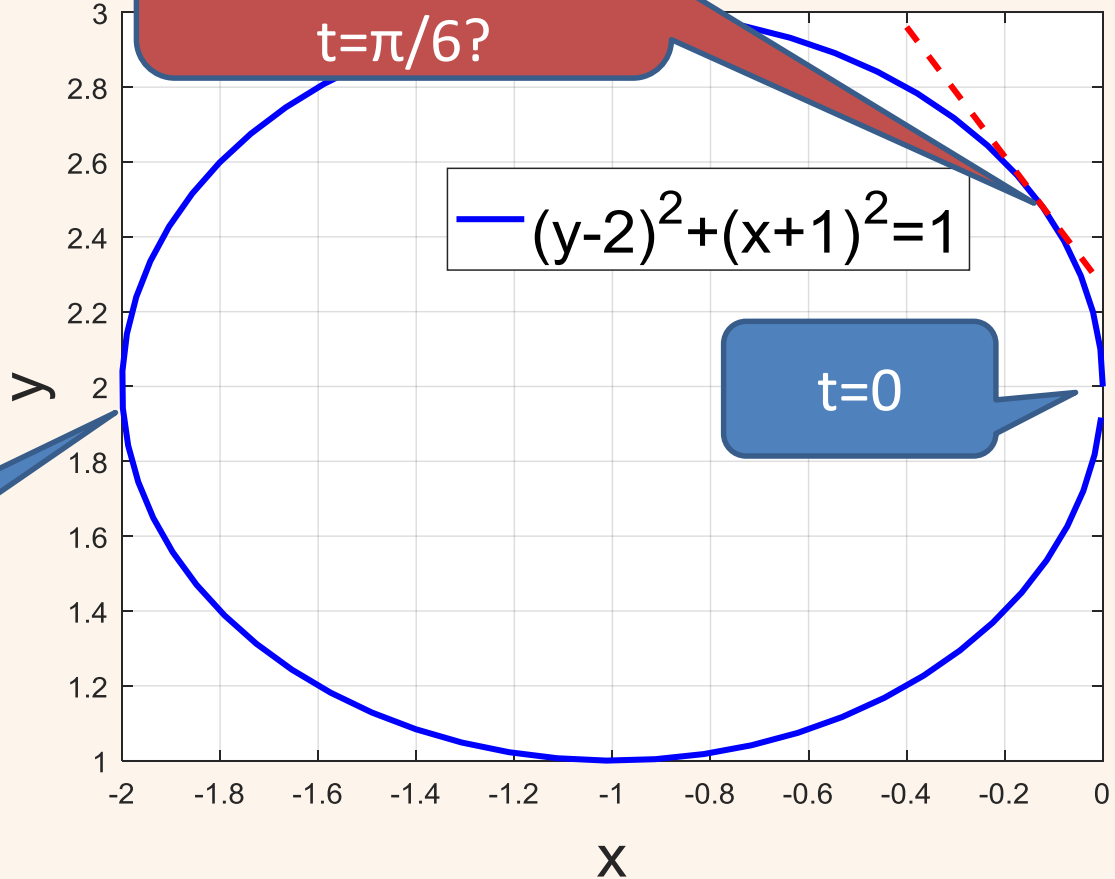
$$y = \sin(t) + 2$$

$t = \pi$

What is the gradient where $t = \pi/6$?

$$(y-2)^2 + (x+1)^2 = 1$$

$t = 0$



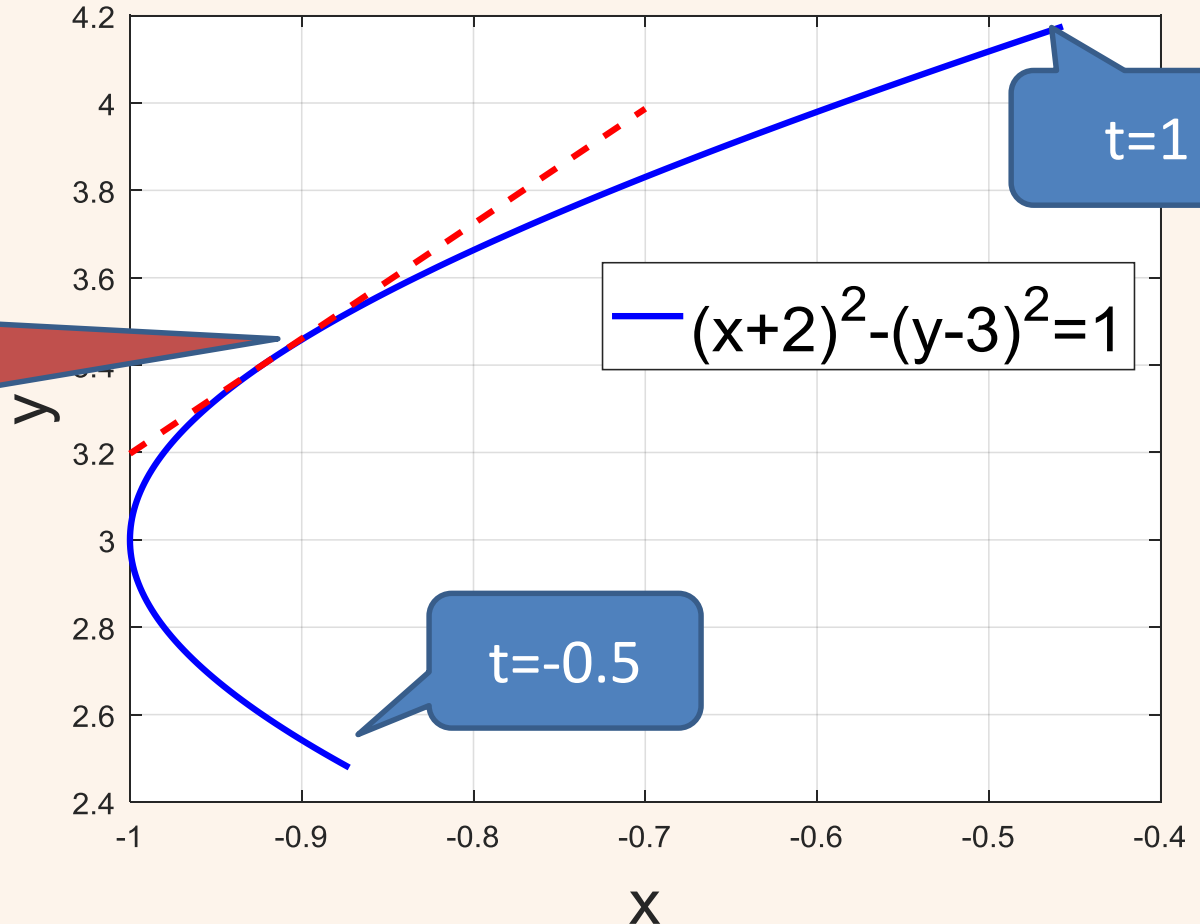
Parametric equations

These are quite common, for example students will be familiar with the equation of a hyperbola.

$$x = \cosh(t) - 2$$

$$y = \sinh(t) + 3$$

What is the gradient where $t=0.4$?

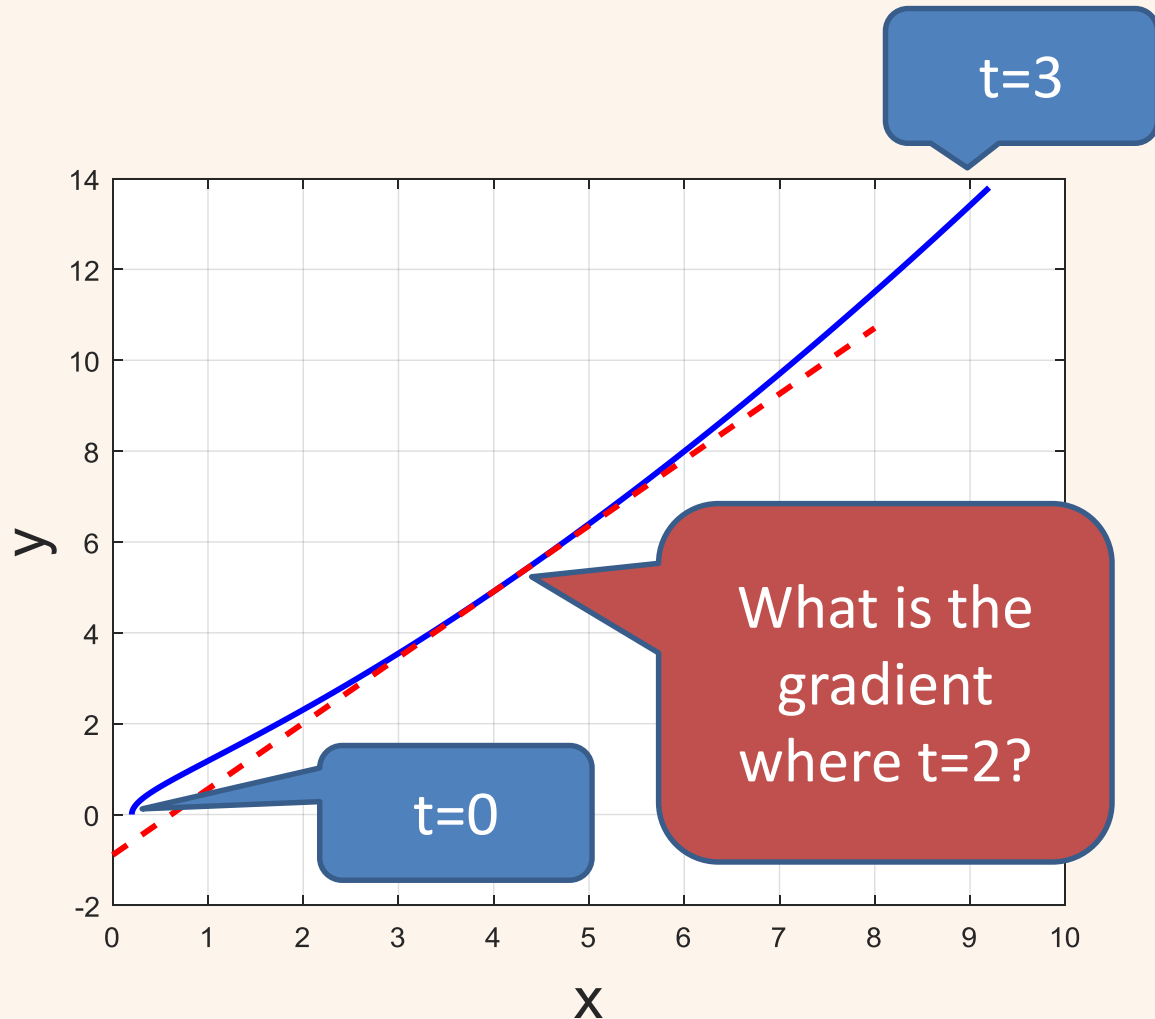


Parametric equations

More challenging examples exist with no underlying simple expression.

$$x = 0.2 + t^2$$

$$y = t + 0.4t^3$$



Differentiation of parametric equations

The basic method here uses the underlying definition of differentiation

$$\frac{dy}{dt} = \lim_{\delta t \rightarrow 0} \frac{y(t + \delta t) - y(t)}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\delta y}{\delta t}$$

$$\frac{dx}{dt} = \lim_{\delta t \rightarrow 0} \frac{x(t + \delta t) - x(t)}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\delta x}{\delta t}$$

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{y(x + \delta x) - y(x)}{(x + \delta x) - x} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta t \rightarrow 0} \frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}} = \frac{dy}{dx}$$

Example 1

Find the gradient of a curve defined with the following parametric equations.

$$x = \cos(t) - 1; \quad y = \sin(t) + 2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

METHOD: Use the formulae

$$\frac{dx}{dt} = -\sin(t); \quad \frac{dy}{dt} = \cos(t)$$

$$\frac{dy}{dx} = \frac{\cos(t)}{-\sin(t)}$$

The answer is in terms of 't'

Example 2

Find the gradient of a curve defined with the following parametric equations.

$$x = \cosh(2t) - 2; \quad y = \sinh(4t) + 3$$

METHOD: Use the formulae

$$\frac{dx}{dt} = 2 \sinh(2t); \quad \frac{dy}{dt} = 4 \cosh(4t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{4 \cosh(4t)}{2 \sinh(2t)}$$

The answer is in terms of 't'

Example 3

Find the gradient of a curve defined with the following parametric equations.

$$x = t + t^2 + \sin(3t); \quad y = t^3 - 1$$

METHOD: Use the formulae

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dx}{dt} = 1 + 2t + 3\cos(3t); \quad \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \frac{3t^2}{1 + 2t + 3\cos(3t)}$$

The answer is in terms of 't'

Summary

- This brief resource has derived the rule for differentiation where a curve is defined through parametric equations.

$$x = f(t); \quad y = g(t)$$

- It shown that the derivative can be defined with the following formulae.

$$\frac{dy}{dx} = \frac{dg/dt}{df/dt} \quad \text{or} \quad \frac{dy/dt}{dx/dt}$$



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