Differentiation 2
first principles

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Introduction

- The previous video introduces the concept of differentiation and the term derivative.
- Next we need to look at how differentiation is performed and the derivative computed.
- The focus here is on 1st principles, that is to show, briefly, how the main results are derived.
- **Students who are happy to go straight to core results without understanding the origins can skip this resource** and go straight to resources 6 to get into some computations.

Recap

- Differentiation means to find the gradient; in general this involves some mathematical operations.
- A derivative is the result of differentiation, that is a function defining the gradient of a curve.
- The **notation** of derivative uses the letter 'd' and **is not a fraction!**

$$y = f(x) \quad \Rightarrow \quad \frac{dy}{dx} = derivative \equiv \frac{df}{dx}$$

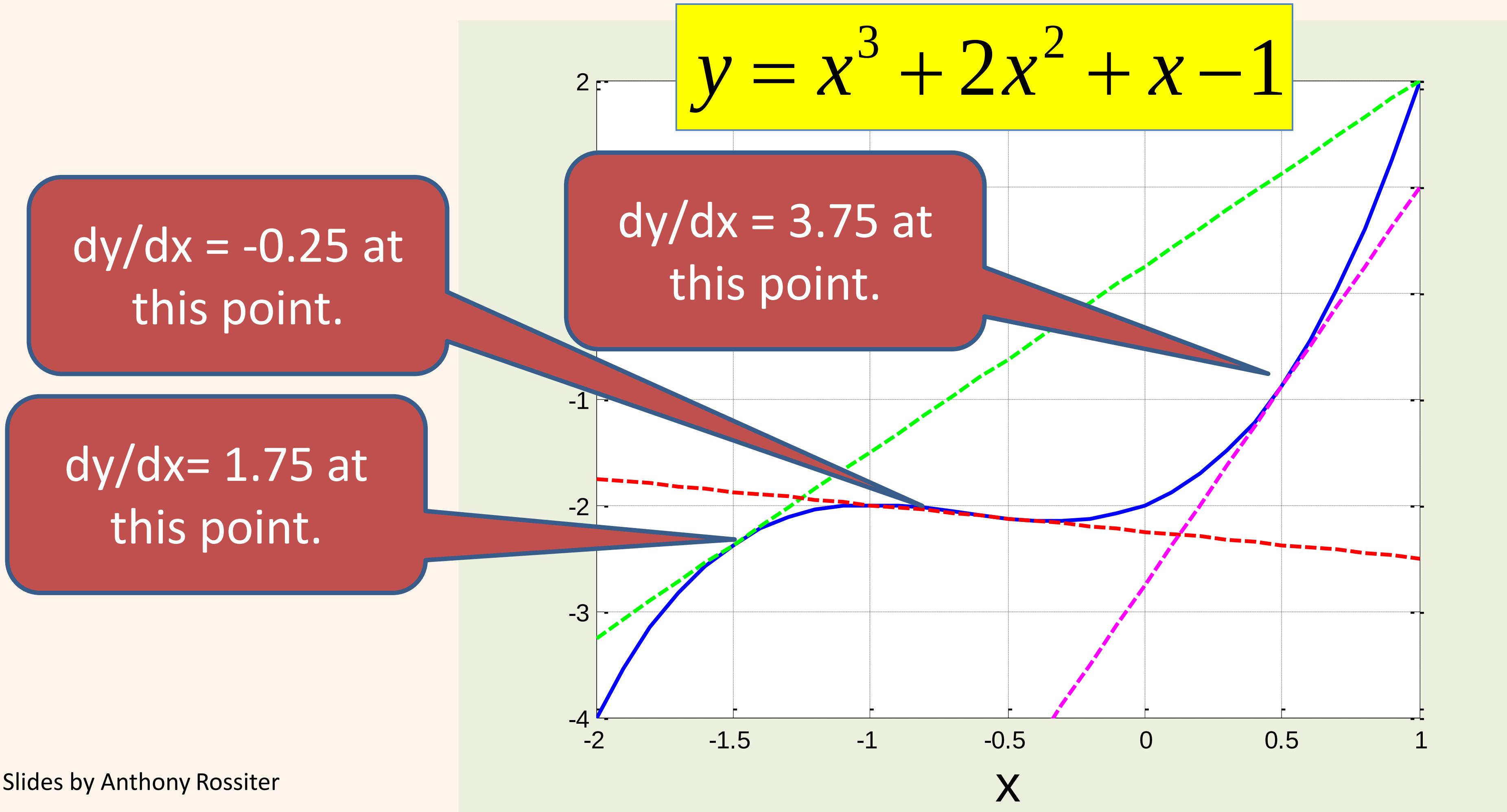
Spoken as 'd f d x'.

$$\frac{d}{dx}(f)$$

Spoken as 'd d x of f'.
The action of differentiation.

What is differentiation?

Differentiation is a process which finds the gradient of a curve, precisely, at any point along the curve.

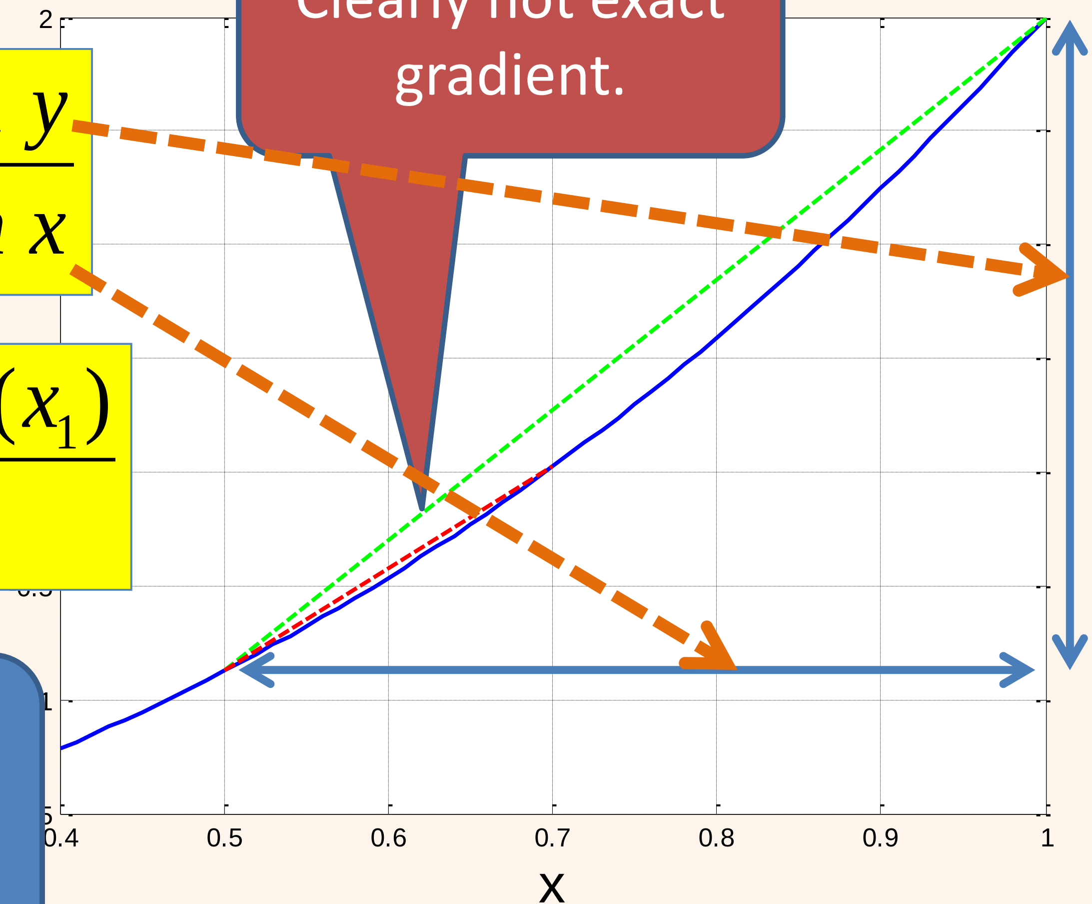


First principles – gradient estimation

For a general curve, the gradient can be estimated using the formulae:

$$gradient \approx \frac{change \;\; in \; y}{change \;\; in \; x}$$

$$gradient \approx \frac{y(x_2) - y(x_1)}{x_2 - x_1}$$



Clearly not exact gradient.

This is close, if difference between the x-values is small.

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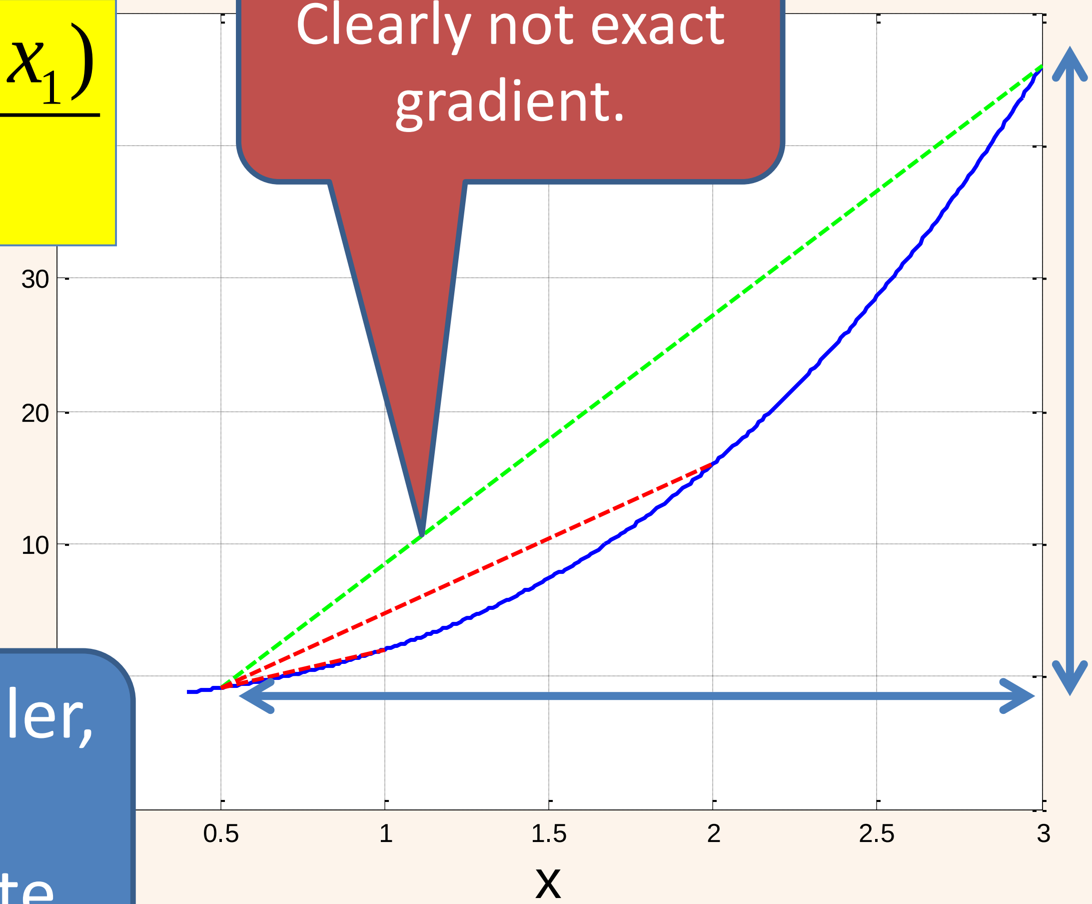


This is close, if difference between the x-values is small.

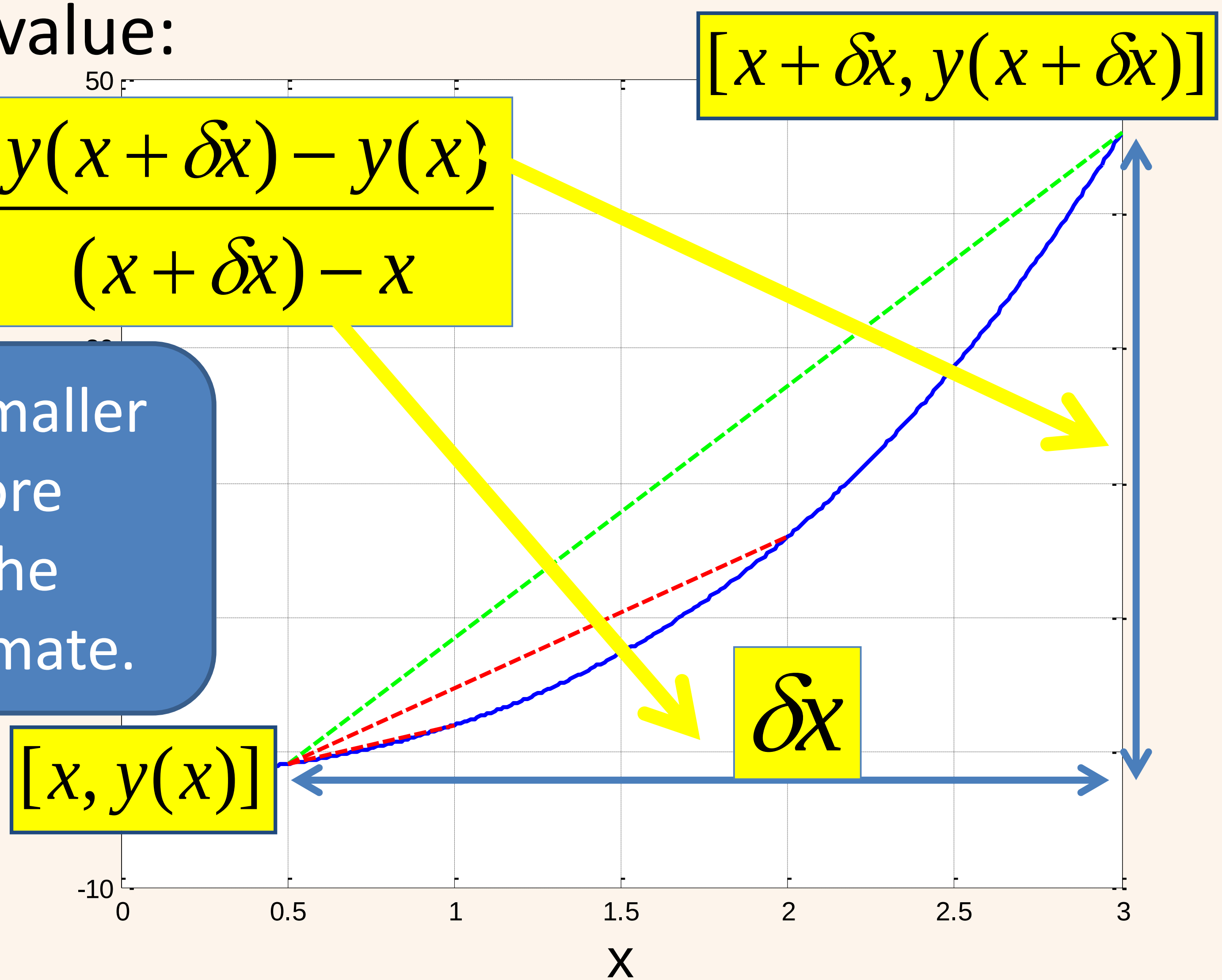
As difference gets smaller, the approximation becomes more accurate.

First principles – gradient estimation

For a general curve, the gradient can be computed as a limiting value:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

Clearly, the smaller δx, the more accurate the gradient estimate.

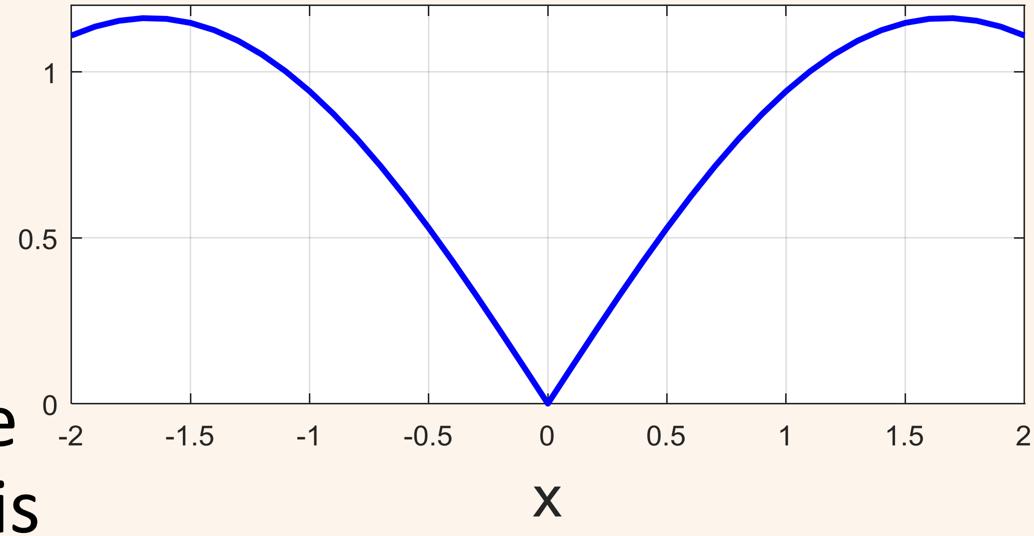


Caviat

We will not dwell on mathematical subtleties, but users need to assume the limit exists and is well defined.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

For many curves, this limit is not unique or well defined at some points and consequently, at those points differentiation is not uniquely defined.



EXAMPLES OF USING FIRST PRINCIPLES TO DERIVE DERIVATIVES OF SOME COMMON FUNCTIONS

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

Example 1 $y = x^2$

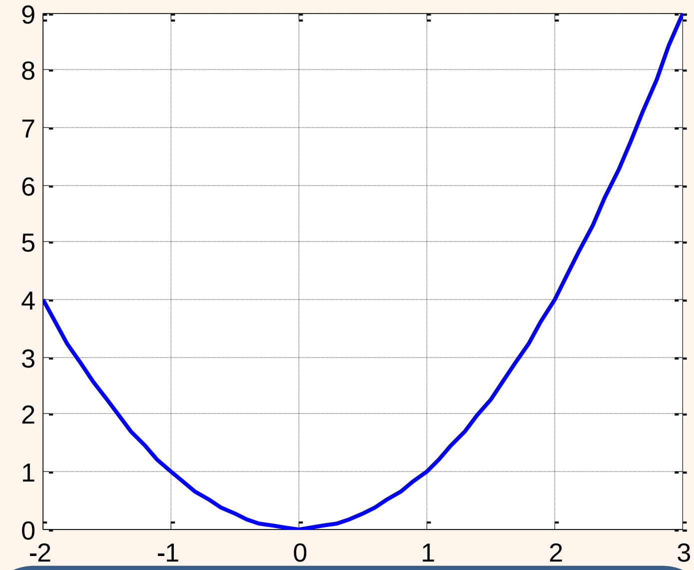
Simply substitute into the formula from the previous page.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{(x+\delta x) - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x^2 + 2x\delta x + (\delta x)^2) - x^2}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(2x + (\delta x))\delta x}{\delta x} = 2x$$



Visual inspection validates this answer is sensible.

Example 2

$$y = x^3$$

Simply substitute into the formula.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^3 - x^3}{(x+\delta x) - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3) - x^3}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(3x^2 + 3x(\delta x) + (\delta x)^2)\delta x}{\delta x} = 3x^2$$

Visual inspection validates this answer is sensible.

Example 3 $y = x^n$

Simply substitute into the formula.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{(x+\delta x) - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x^n + nx^{n-1}\delta x + \cdots) - x^n}{\delta x}$$

Ignore higher order terms in δx as these go to zero.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(nx^{n-1} + \cdots)\delta x}{\delta x} = nx^{n-1}$$

Example 4

$$y = \sin(ax)$$

Simply substitute into the formula.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin(ax + a\delta x) - \sin(ax)}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin(ax)\cos(a\delta x) + \cos(ax)\sin(a\delta x) - \sin(ax)}{\delta x}$$

$$\lim_{\delta x \to 0} \cos(a\delta x) = 1; \quad \lim_{\delta x \to 0} \sin(a\delta x) = a\delta x;$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin(ax) \times 1 + \cos(ax) \times a\delta x - \sin(ax)}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\cos(ax) \times a\delta x}{\delta x} = a\cos(ax)$$

Example 5

$$y = e^{bx}$$

Simply substitute into the formula.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{b(x+\delta x)} - e^{bx}}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{bx}(e^{b\delta x} - 1)}{\delta x}$$

$$\lim_{\delta x \to 0}(e^{b\delta x} - 1) = b\delta x;$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{bx} b\delta x}{\delta x} = be^{bx}$$

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Example 6

$$y = \log ax$$

This one is easiest handled by recognising the following relationship; this is obvious as it amounts to a simple swapping of the axis.

$$\frac{dy}{dx} = \frac{1}{dx/dy} \quad or \quad \frac{dy}{dx}\frac{dx}{dy} = 1$$

$$y = \log(ax) \quad \Rightarrow \quad x = \frac{1}{a}e^{y}$$

$$\frac{dx}{dy} = \frac{1}{a}e^{y} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a}{e^{y}}$$

$$\frac{dy}{dx} = \frac{a}{e^{y}} = \frac{a}{ax} = \frac{1}{x}$$

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Table of some common results

$$y = ax \quad \Rightarrow \quad \frac{dy}{dx} = a$$

$$y = ax^n \quad \Rightarrow \quad \frac{dy}{dx} = nax^{n-1}$$

$$y = \sin(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\cos(bx)$$

$$y = \cos(bx) \quad \Rightarrow \quad \frac{dy}{dx} = -b\sin(bx)$$

$$y = \tan(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\sec^2(bx)$$

$$y = e^{cx} \quad \Rightarrow \quad \frac{dy}{dx} = ce^{cx}$$

$$y = \sinh(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\cosh(bx)$$

$$y = \cosh(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\sinh(bx)$$

$$y = \log x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$

$$y = \frac{1}{\sin(bx)} \quad \Rightarrow \quad \frac{dy}{dx} = -b\frac{1}{\sin(bx)}\cot(x)$$

Summary

- This video has introduced differentiation using first principles derivations.
- The derivatives of a few common functions have been given.
- **Readers can use the same procedures to find derivatives for other functions but in general it is more sensible to access a table of answers which have been derived for you.**
- **Later videos will gradually introduce known formulae and their application.**

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