The University Of Sheffield.

Differentiation 4
product rule explanations

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Introduction

- The previous videos introduced differentiation from first principles.
- It is logical therefore to consider what consequences follow from this derivation.
- The product rule is a scenario where a function comprises the product of other functions.
- How do we find the derivative of a function which is the product of other functions?

Students can skip these explanations and go straight to resources 6 if they want to get straight into some computations.

Product of functions

The scenario we are looking at here is a function which is the product of other functions.

$$y = f(x) = 4x^2 \cos(2x)$$

$$z = g(w) = 3w^{-3} \sin(w/2)$$

$$h = k(t) = 0.2t^2 e^{-0.1t}$$

How do we differentiate such functions?

Core result

The **derivative of a product of TWO functions** is **given by the following formulae**.

$$y = u(x)v(x)$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Staff often verbalise this with the shorthand

"u d v plus v d u"

REMINDER OF DEFINITION OF DERIVATIVE

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

$$\frac{dz}{dw} = \lim_{\delta w \to 0} \frac{z(w+\delta w) - z(w)}{(w+\delta w) - w}$$

Use derivative definition on a product of functions

Let a function be given as follows: $y = u(x)v(x)$

Use 1st principles definition of the derivative.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{u(x+\delta x)v(x+\delta x) - u(x)v(x)}{\delta x}$$

From the definition of derivatives

$$\lim_{\delta x \to 0} u(x+\delta x) = u(x) + \delta x \frac{du}{dx}$$

$$\lim_{\delta x \to 0} v(x+\delta x) = v(x) + \delta x \frac{dv}{dx}$$

Use derivative definition on a product of functions

$$y = u(x)v(x)$$

Substitute in from the previous page:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{u(x+\delta x)v(x+\delta x) - u(x)v(x)}{\delta x}$$

$$u(x+\delta x) = u(x) + \delta x \frac{du}{dx}$$

$$v(x+\delta x) = v(x) + \delta x \frac{dv}{dx}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(u(x) + \delta x \frac{du}{dx})(v(x) + \delta x \frac{dv}{dx}) - u(x)v(x)}{\delta x}$$

Ignore $(\delta x)^2$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{u(x)v(x) + v(x)\delta x \frac{du}{dx} + u(x)\delta x \frac{dv}{dx} - u(x)v(x)}{\delta x}$$

Use derivative definition on a product of functions

$$y = u(x)v(x)$$

Continue from the previous page:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{u(x)v(x) + v(x)\delta x \frac{du}{dx} + u(x)\delta x \frac{dv}{dx} - u(x)v(x)}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\left(v(x)\frac{du}{dx} + u(x)\frac{dv}{dx} \right)\delta x}{\delta x}$$

$$\frac{dy}{dx} = \left(v(x)\frac{du}{dx} + u(x)\frac{dv}{dx} \right)$$

Example 1

Find the derivative of the following.

$$y = f(x) = 4x^2 \cos(2x) = u(x)v(x)$$

$$u = x^2; \quad \frac{du}{dx} = 2x$$

$$v = 4\cos(2x); \quad \frac{dv}{dx} = -8\sin(2x)$$

$$\frac{dy}{dx} = \left(v(x)\frac{du}{dx} + u(x)\frac{dv}{dx} \right)$$

$$\frac{dy}{dx} = 8x\cos(2x) - 8x^2 \sin(2x)$$

Example 2

Find the derivative of the following.

$$z = g(w) = 3w^{-3}\sin(w/2) = u(w)v(w)$$

$$u = 3w^{-3}; \quad \frac{du}{dw} = -9w^{-4}$$

$$v = \sin(w/2); \quad \frac{dv}{dw} = \frac{\cos(w/2)}{2}$$

$$\frac{dz}{dw} = \left(v(w)\frac{du}{dw} + u(w)\frac{dv}{dw} \right)$$

$$\frac{dz}{dw} = -9w^{-4}\sin(w/2) + 1.5w^{-3}\cos(w/2)$$

Summary

- This brief resource has shown that the derivative of a product requires a particular formulae to get simple solutions.
- In other words:

$$y = u(x)v(x)$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

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