The University Of Sheffield.

Differentiation 5 quotient rule explanations

J A Rossiter

Slides by Anthony Rossiter

Introduction

- The previous videos introduced differentiation from first principles.
- It is logical therefore to consider what consequences follow from this derivation.
- The quotient rule is a scenario where a function comprises the division of one function by another.

Students can skip these explanations and go straight to resources 6 if they want to get straight into some computations.

Quotient of functions

The scenario we are looking at here is a function which is one function divided by another function.

$$y = f(x) = \frac{4x^2}{\cos(2x)}$$

$$z = g(w) = 3\frac{\sin(w/2)}{w^3}$$

$$h = k(t) = \frac{0.2t^2}{e^{0.1t}}$$

How do we differentiate such functions?

Core result

The **derivative of a quotient of TWO functions** is **given by the following formulae**.

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Staff often verbalise this with the shorthand

“v d u minus u d v divided by v squared”

REMINDER OF DEFINITION OF DERIVATIVE

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

$$\frac{dz}{dw} = \lim_{\delta w \to 0} \frac{z(w+\delta w) - z(w)}{(w+\delta w) - w}$$

Use derivative definition on a quotient of functions

$$y = \frac{u(x)}{v(x)}$$

Let a function be given as follows:

Use 1st principles definition of the derivative.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{(x+\delta x) - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\frac{u(x+\delta x)}{v(x+\delta x)} - \frac{u(x)}{v(x)}}{\delta x}$$

From the definition of derivatives

$$\lim_{\delta x \to 0} u(x+\delta x) = u(x) + \delta x \frac{du}{dx}$$

$$\lim_{\delta x \to 0} v(x+\delta x) = v(x) + \delta x \frac{dv}{dx}$$

Use derivative definition on a quotient of functions

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\dfrac{u(x+\delta x)}{v(x+\delta x)}-\dfrac{u(x)}{v(x)}}{\delta x}$$

$$\lim_{\delta x\to 0} u(x+\delta x)=u(x)+\delta x\frac{du}{dx}$$

$$\lim_{\delta x\to 0} v(x+\delta x)=v(x)+\delta x\frac{dv}{dx}$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\dfrac{u(x)+\delta x\frac{du}{dx}}{v(x)+\delta x\frac{dv}{dx}}-\dfrac{u(x)}{v(x)}}{\delta x}=\frac{\left(u(x)+\delta x\frac{du}{dx}\right)v(x)-\left(v(x)+\delta x\frac{dv}{dx}\right)u(x)}{(v(x)+\delta x\frac{dv}{dx})v(x)\delta x}$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{v(x)\delta x\frac{du}{dx}-u(x)\delta x\frac{dv}{dx}}{[v(x)]^2\delta x}=\frac{v(x)\frac{du}{dx}-u(x)\frac{dv}{dx}}{[v(x)]^2}$$

Ignore $(\delta x)^2$

Example 1

Find the derivative of the following.

$$y = f(x) = \frac{4x^2}{\cos(2x)} = \frac{u(x)}{v(x)}$$

$$u = 4x^2; \quad \frac{du}{dx} = 8x$$

$$v = \cos(2x); \quad \frac{dv}{dx} = -2\sin(2x)$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{8x\cos(2x) + 8x^2\sin(2x)}{\cos^2 2x}$$

Example 2

Find the derivative of the following.

$$z = g(w) = 3\frac{\sin(w/2)}{w^3} = \frac{u(w)}{v(w)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$u = 3\sin(w/2); \quad \frac{du}{dw} = 3\frac{\cos(w/2)}{2}$$

$$v = w^3; \quad \frac{dv}{dw} = 3w^2$$

$$\frac{dz}{dw} = \frac{1.5w^3\cos(w/2) - 9w^2\sin(w/2)}{w^6}$$

Summary

- This brief resource has shown that the derivative of a quotient requires a particular formulae to get simple solutions.
- In other words:

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The University Of Sheffield.

Anthony Rossiter
Department of Automatic Control and Systems Engineering
University of Sheffield
www.shef.ac.uk/acse