



# Differentiation 5

## quotient rule explanations

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# Introduction

- The previous videos introduced differentiation from first principles.
- It is logical therefore to consider what consequences follow from this derivation.
- The quotient rule is a scenario where a function comprises the division of one function by another.

Students can skip these explanations and go straight to resources 6 if they want to get straight into some computations.

# Quotient of functions

The scenario we are looking at here is a function which is one function divided by another function.

$$y = f(x) = \frac{4x^2}{\cos(2x)}$$

$$z = g(w) = 3 \frac{\sin(w/2)}{w^3}$$

$$h = k(t) = \frac{0.2t^2}{e^{0.1t}}$$

How do we differentiate such functions?

# Core result

The **derivative of a quotient of TWO functions** is given by the following formulae.

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Staff often verbalise this with the shorthand

“v d u minus u d v divided by v squared”

# REMINDER OF DEFINITION OF DERIVATIVE

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{y(x + \delta x) - y(x)}{(x + \delta x) - x}$$

$$\frac{dz}{dw} = \lim_{\delta w \rightarrow 0} \frac{z(w + \delta w) - z(w)}{(w + \delta w) - w}$$

# Use derivative definition on a quotient of functions

$$y = \frac{u(x)}{v(x)}$$

Let a function be given as follows:

Use 1<sup>st</sup> principles definition of the derivative.

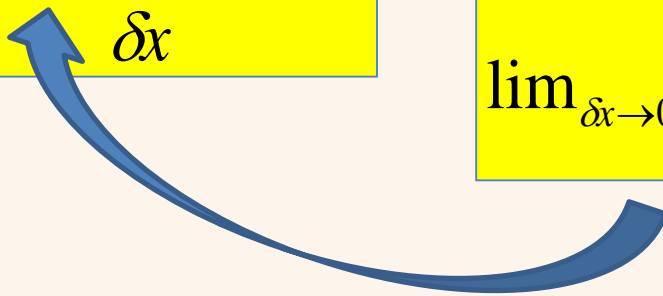
$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{y(x + \delta x) - y(x)}{(x + \delta x) - x}$$

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\frac{u(x + \delta x)}{v(x + \delta x)} - \frac{u(x)}{v(x)}}{\delta x}$$

From the definition of derivatives

$$\lim_{\delta x \rightarrow 0} u(x + \delta x) = u(x) + \delta x \frac{du}{dx}$$

$$\lim_{\delta x \rightarrow 0} v(x + \delta x) = v(x) + \delta x \frac{dv}{dx}$$



# Use derivative definition on a quotient of functions

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\frac{u(x + \delta x)}{v(x + \delta x)} - \frac{u(x)}{v(x)}}{\delta x}$$

$$\lim_{\delta x \rightarrow 0} u(x + \delta x) = u(x) + \delta x \frac{du}{dx}$$

$$\lim_{\delta x \rightarrow 0} v(x + \delta x) = v(x) + \delta x \frac{dv}{dx}$$

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\frac{u(x) + \delta x \frac{du}{dx}}{v(x) + \delta x \frac{dv}{dx}} - \frac{u(x)}{v(x)}}{\delta x} = \frac{\left(u(x) + \delta x \frac{du}{dx}\right)v(x) - \left(v(x) + \delta x \frac{dv}{dx}\right)u(x)}{(v(x) + \delta x \frac{dv}{dx})v(x)\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{v(x)\delta x \frac{du}{dx} - u(x)\delta x \frac{dv}{dx}}{[v(x)]^2 \delta x} = \frac{v(x) \frac{du}{dx} - u(x) \frac{dv}{dx}}{[v(x)]^2}$$

Ignore  $(\delta x)^2$

# Example 1

Find the derivative of the following.

$$y = f(x) = \frac{4x^2}{\cos(2x)} = \frac{u(x)}{v(x)}$$

$$u = 4x^2; \quad \frac{du}{dx} = 8x$$

$$v = \cos(2x); \quad \frac{dv}{dx} = -2\sin(2x)$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{8x \cos(2x) + 8x^2 \sin(2x)}{\cos^2 2x}$$



# Example 2

Find the derivative of the following.

$$z = g(w) = 3 \frac{\sin(w/2)}{w^3} = \frac{u(w)}{v(w)}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$u = 3 \sin(w/2); \quad \frac{du}{dw} = 3 \frac{\cos(w/2)}{2}$$

$$v = w^3; \quad \frac{dv}{dw} = 3w^2$$

$$\frac{dz}{dw} = \frac{1.5w^3 \cos(w/2) - 9w^2 \sin(w/2)}{w^6}$$

# Summary

- This brief resource has shown that the derivative of a quotient requires a particular formulae to get simple solutions.
- In other words:

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$



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