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Differentiation 9
examples using the product and quotient rules

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http://controleducation.group.shef.ac.uk/mathematics.html

Introduction

- The previous videos have given a definition and concise derivation of differentiation from first principles.
- The aim now is to give a number of worked examples for more challenging cases.
- Here the focus is on combining the product and quotient rules, while also utilising a table of results for simple functions.

$$y = u(x)v(x)$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Table of common results

$$y = ax \quad \Rightarrow \quad \frac{dy}{dx} = a$$

$$y = ax^n \quad \Rightarrow \quad \frac{dy}{dx} = nax^{n-1}$$

$$y = \sin(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\cos(bx)$$

$$y = \cos(bx) \quad \Rightarrow \quad \frac{dy}{dx} = -b\sin(bx)$$

$$y = \tan(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\sec^2(bx)$$

$$y = e^{cx} \quad \Rightarrow \quad \frac{dy}{dx} = ce^{cx}$$

$$y = \sinh(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\cosh(bx)$$

$$y = \cosh(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\sinh(bx)$$

$$y = \log x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$

$$y = \cos ec(bx) \quad \Rightarrow \quad \frac{dy}{dx} = -b\frac{\cos(x)}{\sin^2(bx)}$$

$$y = \cot(bx) \quad \Rightarrow \quad \frac{dy}{dx} = -b\cos ec^2(bx)$$

$$y = \sec(bx) \quad \Rightarrow \quad \frac{dy}{dx} = b\frac{\sin(x)}{\cos^2(bx)}$$

NUMERICAL EXAMPLES

KEY TECHNIQUES

1. Define all functions used in the product and quotient rules, with their associated derivatives, clearly.
2. Ensure the layout of the work is uncluttered and unambiguous. This will avoid many typos.
3. Use known results from a table wherever possible.

Example 1

Find the derivative of:

$$y = f(x) = \frac{x^2 \log(2x^3)}{(x^5+4x+1)} = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$u = x^2(\log 2 + 3\log x) = p(x)t(x);$$

$$\frac{du}{dx} = p\frac{dt}{dx} + t\frac{dp}{dx}$$

Here u(x) is a product of two functions, so we need the product rule to differentiate this.

$$p(x) = x^2; \quad t(x) = (\log 2 + 3\log x);$$

$$\frac{dp}{dx} = 2x; \quad \frac{dt}{dx} = \frac{3}{x}$$

$$\frac{du}{dx} = 2x(\log 2 + 3\log x) + x^2\frac{3}{x}$$

Example 1 - continued

Find the derivative of:

$$y = f(x) = \frac{x^2 \log(2x^3)}{(x^5+4x+1)} = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{du}{dx} = 2x(\log 2 + 3\log x) + x^2\frac{3}{x}$$

$$v = x^5 + 4x + 1;$$

$$\frac{dv}{dx} =$$

Substitute into quotient formulae

Example 2

Find the derivative of:

$$h = g(w) = \frac{4e^{2w}}{w^2 \sec(3w)} = \frac{u(w)}{v(w)}$$

$$\frac{dh}{dw} = \frac{v\frac{du}{dw} - u\frac{dv}{dw}}{v^2}$$

$$v = w^2 \sec(3w) = p(w)t(w);$$

$$\frac{dv}{dw} = p\frac{dt}{dw} + t\frac{dp}{dw}$$

Here v(w) is a product of two functions, so we need the product rule to differentiate this.

$$p(w) = w^2; \quad t(w) = \sec(3w);$$

$$\frac{dp}{dw} = 2w; \quad \frac{dt}{dw} = \frac{3\sin(3w)}{\cos^2(3w)}$$

Straight from the table of known results.

Example 2 - continued

Find the derivative of:

$$h = g(w) = \frac{4e^{2w}}{w^2 \sec(3w)} = \frac{u(w)}{v(w)}$$

$$\frac{dh}{dw} = \frac{v\frac{du}{dw} - u\frac{dv}{dw}}{v^2}$$

$$\frac{dv}{dw} = w^2 \frac{3\sin(3w)}{\cos^2(3w)} + 2w\sec(3w);$$

$$u = 4e^{2w}; \quad \frac{du}{dw} = 8e^{2w};$$

From previous page.

Straight from the table of known results.

Substitute into quotient formulae

Example 3

Find the derivative of:

$$\frac{dh}{dw} = \frac{v\frac{du}{dw} - u\frac{dv}{dw}}{v^2}$$

$$h = g(w) = \frac{6\sin(0.5w)\cos(0.5w)\log(3w)}{e^{-w}\tan(0.2w)} = \frac{u(w)}{v(w)}$$

Here u(w) is a product of **three** functions and v(w) is a product of two functions, so we need the product rule for both.

However, using tables of known results, students will see a possible double angle formulae in the numerator which will simplify the overall function.

$$h = g(w) = \frac{3\sin(w)\log(3w)}{e^{-w}\tan(0.2w)} = \frac{u(w)}{v(w)}$$

Next, use product rule to find derivatives of u(w) and v(w).

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Example 3

$$h = g(w) = \frac{3\sin(w)\log(3w)}{e^{-w}\tan(0.2w)} = \frac{u(w)}{v(w)}$$

Find derivatives of u(w) and v(w) using the product rule.

$$u = 3\sin(w)\log(3w) = p(w)t(w);$$

$$\frac{du}{dw} = p\frac{dt}{dw} + t\frac{dp}{dw}$$

$$p(w) = 3\sin(w); \quad t(w) = \log(3w);$$

$$\frac{dp}{dw} = 2\cos(w); \quad \frac{dt}{dw} = \frac{1}{w}$$

Straight from the table of known results.

$$v = e^{-w}\tan(0.2w) = q(w)r(w);$$

$$\frac{dv}{dw} = q\frac{dr}{dw} + r\frac{dq}{dw}$$

$$q(w) = e^{-w}; \quad r(w) = \tan(0.2w);$$

$$\frac{dq}{dw} = -e^{-w}; \quad \frac{dr}{dw} = 0.2\sec^2(0.2w)$$

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Example 3

$$h = g(w) = \frac{3\sin(w)\log(3w)}{e^{-w}\tan(0.2w)} = \frac{u(w)}{v(w)}$$

Using results of previous page.

$$\frac{du}{dw} = 3\sin(w)\frac{1}{w} + 2\cos(w)\log(3w)$$

$$\frac{dv}{dw} = e^{-w}0.2\sec^2(0.2w) - \tan(0.2w)e^{-w}$$

Finally, substitute into quotient formulae.

$$\frac{dh}{dw} = \frac{v\frac{du}{dw} - u\frac{dv}{dw}}{v^2}$$

Summary

- This video has demonstrated the differentiation of commonplace functions using a lookup table in combination with the product and quotient rules.
- Viewers will see that the most important points are:
  - Keep clear definitions of functions used in the product and quotient rules and their derivatives before substituting into the formulae.
  - Use a lookup table for common results.
  - Don’t worry if the algebra gets messy, but make sure the layout is clear and well organised.

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