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# Matrices 10: 3x3 determinants

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**http://controleducation.group.shef.ac.uk/indexwebbook.html**

**http://www.shef.ac.uk/acse**

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# Introduction

- This video looks at the concepts of a determinant.
- The previous video introduced the definition for 2 by 2 matrices.
- This video introduces definitions for 3x3 matrices.

VIEWERS should note that a determinant is a **DEFINITION** – they cannot be proved or derived.

**Also, a determinant is only defined for square matrices.**

# Determinant definition for 2 by 2 matrices

The determinant is defined as product of the diagonal elements minus the product of the off-diagonal elements.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad |A| = ad - bc$$

Note use of vertical lines around matrix is notation used to define determinant.

# Determinants of higher dimension matrices

- This definition is move complicated and requires a number of other definitions to be clarified first.
- Terms such as minors and cofactors are needed.

# Determinant for 3 by 3 matrices

We will not prove that all these formulae give the same result although that is key to all the following.

Next, **by definition**:

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$or = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$or = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

SUM along any row

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$
$$or = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$
$$or = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

SUM along any column

# Cofactors

- We need the cofactors to find the determinant.
- However the definition of cofactors requires another definition, that is of the minors which is introduced next.

# Defining a minor for 3 by 3 matrices

A minor (here use notation $a_{ij}^{M}$) for a given coefficient is taken as the matrix with the relevant row and column removed.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad a_{13}^{M} =$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad a_{22}^{M} =$$

Find the minors for positions {1,2},{2,3},{3,1}

$$A = \begin{bmatrix} 3 & 6 & 5 \\ 8 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

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# Minors

Minors are still matrices.

We define minors for each coefficient thus a 3x3 matrix will have 9 minors, each of which is a matrix in its self.

# Chess board of signs

Before defining cofactors, we need the concept of signs linked to matrix position.

The sign matrix begins with a '+' in the {1,1} position.

Neighbouring terms have opposite signs.

$$sign = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix};$$

Position {i,j} is + if i+j is even

Position {i,j} is - if i+j is odd

# Defining the cofactor terms

A cofactor is taken as the determinant of a minor with a sign taken from the sign matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad a_{13}^{M} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$sign = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix};$$

$$a_{21}^{M} = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

$$A_{13} = +\left| a_{13}^{M} \right| = a_{21}a_{32} - a_{22}a_{31}$$

$$A_{21} = -\left| a_{21}^{M} \right| = -(a_{12}a_{33} - a_{13}a_{32})$$

# Determinant for 3 by 3 matrices

The determinant is defined using the concept of a matrix of cofactors $A_{ij}$. For convenience define the original matrix with lower case and the matrix of cofactors with upper case.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad cof(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Expand along any row or column:

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$or = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$or = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

SUM along any row

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$
$$or = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$
$$or = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

SUM along any column

# Find the determinant using the top row expansion

$$A = \begin{bmatrix} 3 & 6 & 5 \\ 8 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

# Find the determinant using the 2nd column expansion

$$A = \begin{bmatrix} 0 & 3 & 4 \\ -2 & -1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

$$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

# Find the determinant using an expansion of your choice

$$A = \begin{bmatrix} 0 & -2 & 4 \\ 0 & 8 & 2 \\ 2 & 1 & 6 \end{bmatrix}$$

# Summary

1. Defined determinant for 3x3 matrices.
2. The definition required further definitions.
   - The concept of a minor
   - The concept of a sign matrix.
   - The concept of a cofactor.
3. Demonstrated that selecting the row or column for the expansion judiciously can reduce computational effort.
4. Nevertheless, in general require 3 (2x2) determinants which is tedious.