The University Of Sheffield.

Matrices 11: determinants of large matrices

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http://controleducation.group.shef.ac.uk/indexwebbook.html

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Introduction

- Previous videos introduced the concepts of a determinant and definitions for 2x2 and 3x3 matrices.
- This video introduces a generic definition for large dimension matrices.

VIEWERS should note that a determinant is a **DEFINITION** – they cannot be proved or derived.

Also, a determinant is only defined for square matrices.

Determinant definition for 2 by 2 matrices

The determinant is defined as product of the diagonal elements minus the product of the off-diagonal elements.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad |A| = ad - bc$$

Determinant for 3 by 3 matrices

We will not prove that all these formulae give the same result although that is key to all the following.

Next, **by definition**:

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$or = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$or = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

SUM along any row

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$
$$or = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$
$$or = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

SUM along any column

We will not prove that all these formulae give the same result although that is key to all the following.

Next, by definition:

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14}$$

$$\text{or} \quad \sum_{j=1}^{n} a_{ij}A_{ij}$$

More general formulae allow expansion along any row or column and hence:

$$\text{or} \quad \sum_{i=1}^{n} a_{ij}A_{ij}$$

The extension to larger matrices should be obvious, although this is not a paper and pen exercise.

Remarks

- We need to give a general definition for cofactors.
- Within this we need a general definition of a sign matrix and a general definition for a minor.
- These are analogous to those used for 3x3 matrices in previous video so will be covered briefly.

Defining a minor for 4 by 4 matrices

A minor (here use notation a_{ij}^{M}) for a given coefficient is taken as the matrix with the relevant row and column removed.

For example, to find the minor a_{13}^{M} remove the 1st row and 3rd column.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$a_{13}^{M} = \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix};$$

An equivalent definition applies for higher dimensional matrices

Chess board of signs

Before defining cofactors, we need the concept of signs linked to matrix position.

The sign matrix begins with a ‘+’ in the {1,1} position.

Neighbouring terms have opposite signs.

$$sign = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \quad or \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad or \quad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

{i,j} is + if i+j is even

{i,j} is - if i+j is odd

Definition for larger matrices is equivalent

Defining the cofactor terms for a 4 by 4

A cofactor is taken as the determinant of a minor with a sign change if appropriate to the position.

Find the cofactor A_{23}.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

First define minor

$$a_{23}^{M} = \begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}$$

Then cofactor is determinant of minor with appropriate sign (Here – as 2+3 is odd).

$$A_{23} = -\left|a_{23}^{M}\right| = -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

Computational load for a 4x4 determinant

To complete the cofactor matrix, we need to find 16 3x3 minors followed by their determinants.

$$|A| = \begin{bmatrix} \left|a^M_{1,1}\right| & -\left|a^M_{1,2}\right| & \left|a^M_{1,3}\right| & -\left|a^M_{1,4}\right| \\ -\left|a^M_{2,1}\right| & \left|a^M_{2,2}\right| & -\left|a^M_{2,3}\right| & \left|a^M_{2,4}\right| \\ \left|a^M_{3,1}\right| & -\left|a^M_{3,2}\right| & \left|a^M_{3,3}\right| & -\left|a^M_{3,4}\right| \\ -\left|a^M_{4,1}\right| & \left|a^M_{4,2}\right| & -\left|a^M_{4,3}\right| & \left|a^M_{4,4}\right| \end{bmatrix}$$

Each 3x3 determinant computation requires 3 2x2 determinant computations (as we use just a single row or column of cofactors).

Remarks on determinants of n x n matrices

A minor is now a (n-1) by (n-1) matrix and therefore to form the cofactor matrix requires:

(n x n) determinants where the determinants are dimension (n-1)x(n-1).

For example a determinant of a (5x5):

- Requires 5 (4x4) determinants.
- Each (4x4) requires 4 (3x3) determinants.
- Each (3x3) requires 3 (2x2) determinants.

5x4x3 (2x2) determinants

Determinants of larger matrices cannot be handled with a simple application of the definition as the number of computations quickly becomes unmanageable.

Numerical examples

We are not even going to bother doing a hard numerical example of a full 4x4 determinant because this is so tedious.

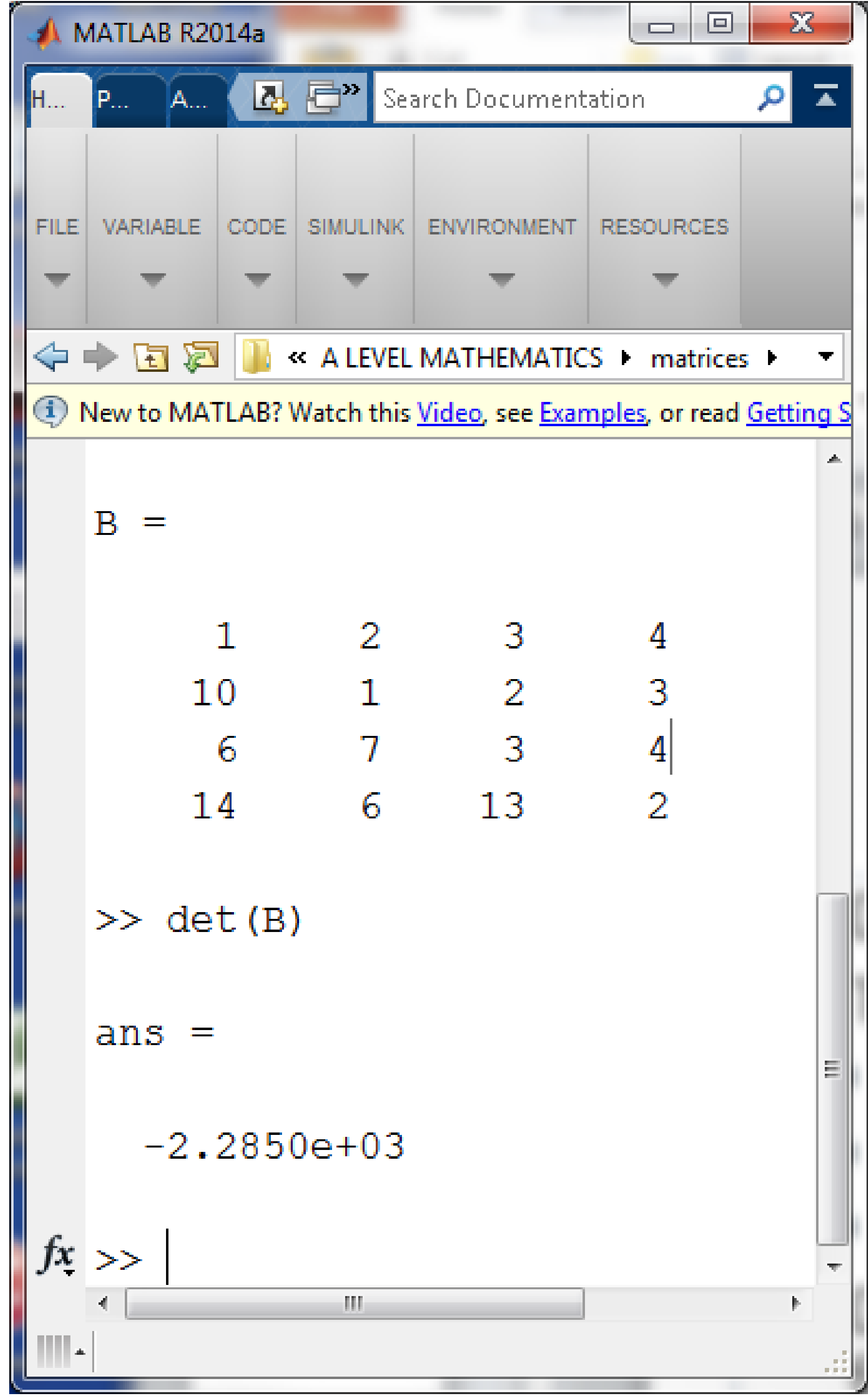
You can of course get answers from MATLAB, but that tool will use **shortcuts and tricks** covered in the next video.

HOWEVER; Some cases can be done by inspection and are given next.

Find the determinant (using MATLAB)

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 1 & 2 & 3 \\ 6 & 7 & 3 & 4 \\ 14 & 6 & 13 & 2 \end{bmatrix}$$

Main file is det.m



Triangular matrices (3x3)

Triangular matrices (thus with just zeros in the upper or lower triangle) allow obvious and simple shortcuts to find the determinant.

Basically use the product of the diagonal elements

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & 7 & 8 \end{bmatrix}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$or = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$or = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} + 0 \times A_{12} + 0 \times A_{13} = a_{11}a_{22}a_{33} = 1 \times (-1) \times 8$$

Triangular matrices (4x4)

Determinant of 4x4 triangular matrices is the product of the diagonal elements

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 22 & 2 & 0 & 0 \\ 5 & 18 & 3 & 0 \\ 104 & 56 & 13 & 2 \end{bmatrix}$$

$$|B| = b_{11}B_{11} + b_{12}B_{12} + b_{13}B_{13} + b_{14}B_{14}$$

NEXT use result for 3x3 from previous slide.

$$|B| = b_{11}\begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{vmatrix} + 0B_{12} + 0B_{13} + 0B_{14} = b_{11}\begin{vmatrix} b_{22} & 0 & 0 \\ b_{32} & b_{33} & 0 \\ b_{42} & b_{43} & b_{44} \end{vmatrix}$$

Find the determinant

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 6 & -4 & 0 \\ 4 & 26 & -8 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 8 & 4 & 0 \\ 0 & 2 & 6 & -9 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Find the determinant using judicious choice of row or column for expansion

$$B = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 3 & -12 & -5 & 0 \\ 5 & 0 & 0 & 0 \\ 104 & 56 & 13 & 2 \end{bmatrix}$$

EXPAND ALONG 3rd row
(or 4th column)

$$|B| = 5\begin{vmatrix} 2 & 0 & 0 \\ -12 & -5 & 0 \\ 56 & 13 & 2 \end{vmatrix} + 0B_{32} + 0B_{33} + 0B_{34} = 5 \times 2 \times (-5) \times 2$$

Summary

- Defined determinant for n-dimensional square matrices using cofactors (and minors).
- Expansion can be along any row or column.
- The underlying definition is very tedious to compute and NOT a paper exercise in general for matrices (4x4) and larger.
- Shortcuts exist for upper and lower triangular matrices and **sometimes** with sparse matrices it is possible to identify an easy solution.