The University Of Sheffield.

Matrices 12: efficient determinant computation

Anthony Rossiter

http://controleducation.group.shef.ac.uk/indexwebbook.html

http://www.shef.ac.uk/acse

Department of Automatic Control
and Systems Engineering

Introduction

- Previous videos introduced the concepts of a determinant but it was clear that in general these would be rather tedious to compute.
- This introduces rules and shortcuts which allow much faster and easier computation.
- For a matrix with coefficients a_{ij} and cofactors A_{ij}, determinant is defined from expansion along any row or column, that is:

$$\sum_{j=1}^{n} a_{ij} A_{ij} \qquad \sum_{i=1}^{n} a_{ij} A_{ij}$$

Properties of determinants

Properties of determinants are critically to understanding how we can manipulate a given matrix to simplify the determinant computations.

Several properties will be demonstrated and then exploited over the next few videos.

This video focuses on two elementary properties linked to scaling of coefficients.

IMPORTANT REMINDER

1. It was noted in the previous video that if a matrix is upper or lower triangular, then the determinant reduces to the product of the diagonal elements.
2. It was also noted that for sparse matrices, the judicious choice of row or column for the expansion will significantly reduce the overall computation.

THE RULES developed often exploit those two observations by attempting to rearrange a matrix into a form where sparsity can be exploited.

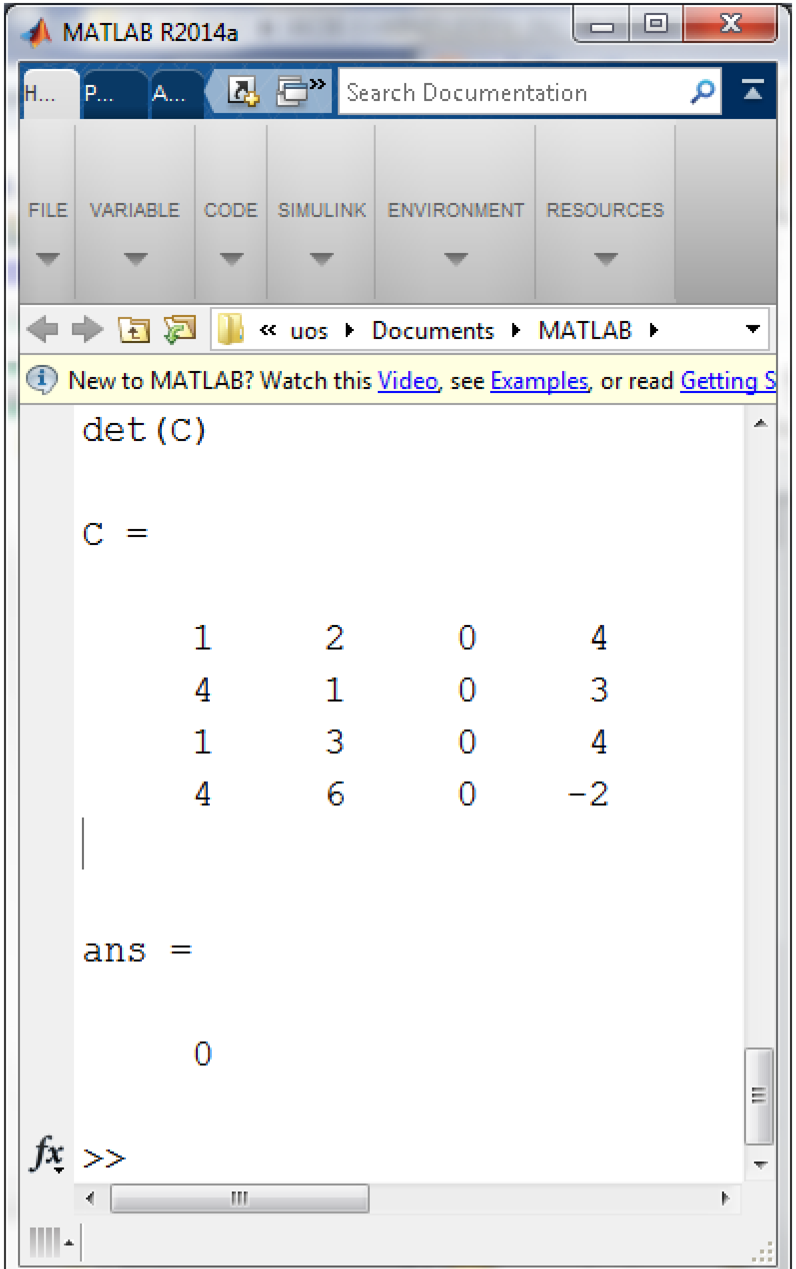
Observation

If a matrix has a entire row or column of zeros then the determinant is zero.

This should be obvious from expansion along the given row/column.

$$\left\{a_{ij} = 0, \forall i\right\} \;\Rightarrow\; \sum_{i=1}^{n} a_{ij} A_{ij} = 0$$

MATLAB example



SCALING ANY ROW (OR COLUMN) BY K RESULTS IN A SCALING OF THE DETERMINANT BY K

Scaling any row (or column) by λ results in a scaling of the determinant by λ.

First illustrate what we mean by scaling any row or column.

Here scale the 2[nd] row.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix} \Rightarrow B = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ \lambda a_{2,1} & \lambda a_{2,2} & \lambda a_{2,3} & \lambda a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 6 & -1 & 0 \\ 4 & 7 & 8 \end{bmatrix} \quad \Rightarrow \quad B = \begin{bmatrix} 1 & 0 & 3 \\ \lambda 6 & -\lambda & 0 \\ 4 & 7 & 8 \end{bmatrix}$$

Cofactors of 2nd row of B are the same as cofactors of the 2nd row of A.

If the determinant expansion is done along the same row as the scaling, the cofactors are unaffected.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix} \Rightarrow B = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ \lambda a_{2,1} & \lambda a_{2,2} & \lambda a_{2,3} & \lambda a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24}$$

$$|B| = \lambda a_{21}A_{21} + \lambda a_{22}A_{22} + \lambda a_{23}A_{23} + \lambda a_{24}A_{24}$$

Scaling any row (or column) by λ results in a scaling of the determinant by λ.

This follows directly from the definition.

$$|A| = \sum_{j=1}^{4} a_{ij} A_{ij} \qquad or \quad \sum_{i=1}^{4} a_{ij} A_{ij}$$

Let the scaling be on the **row/column** used for the expansion **so that cofactors are unaffected**:

HENCE:

$$|B| = \sum_{j=1}^{4} (\lambda a_{ij}) A_{ij} = \lambda \sum_{j=1}^{4} a_{ij} A_{ij} = \lambda |A|$$

Numerical example

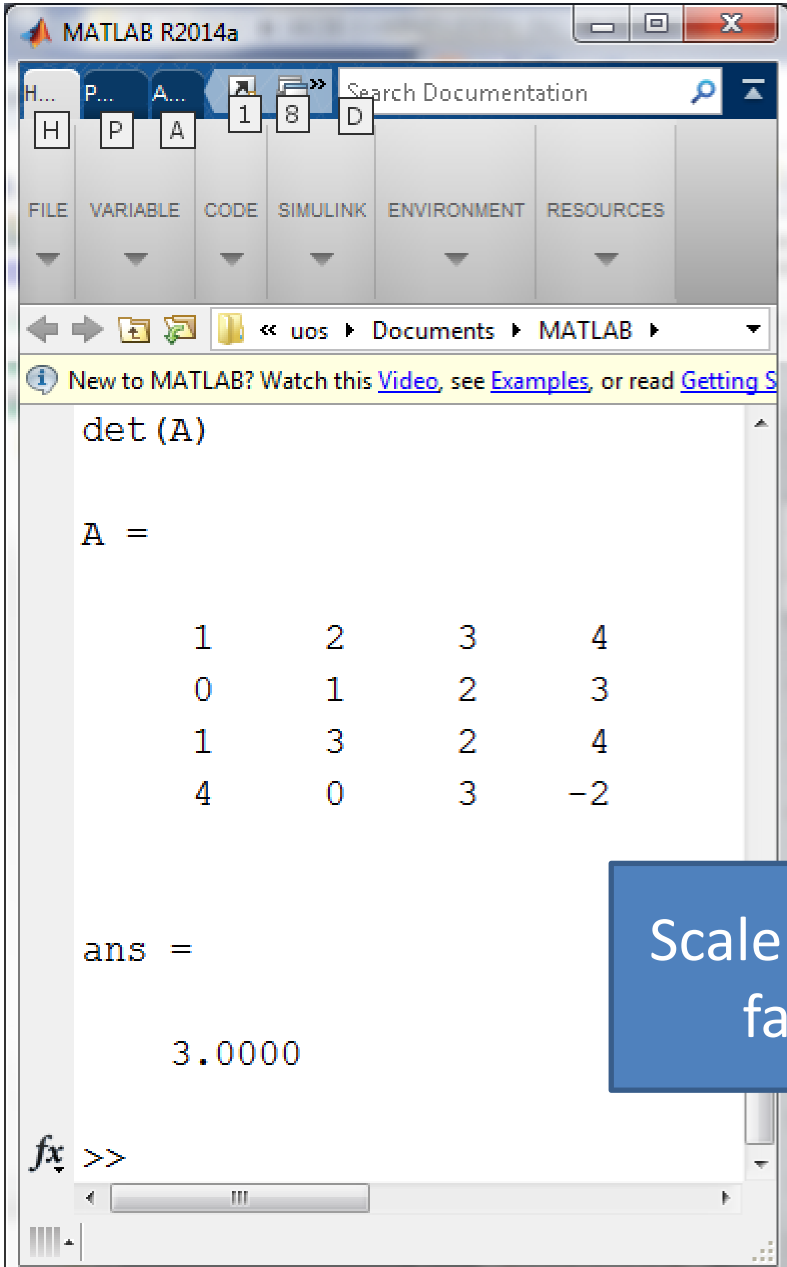
$$A = \begin{bmatrix} 1 & 0 & 3 \\ 6 & -1 & 0 \\ 4 & 7 & 8 \end{bmatrix} \quad \Rightarrow \quad B = \begin{bmatrix} 1 & 0 & 3 \\ \lambda 6 & -\lambda & 0 \\ 4 & 7 & 8 \end{bmatrix}$$

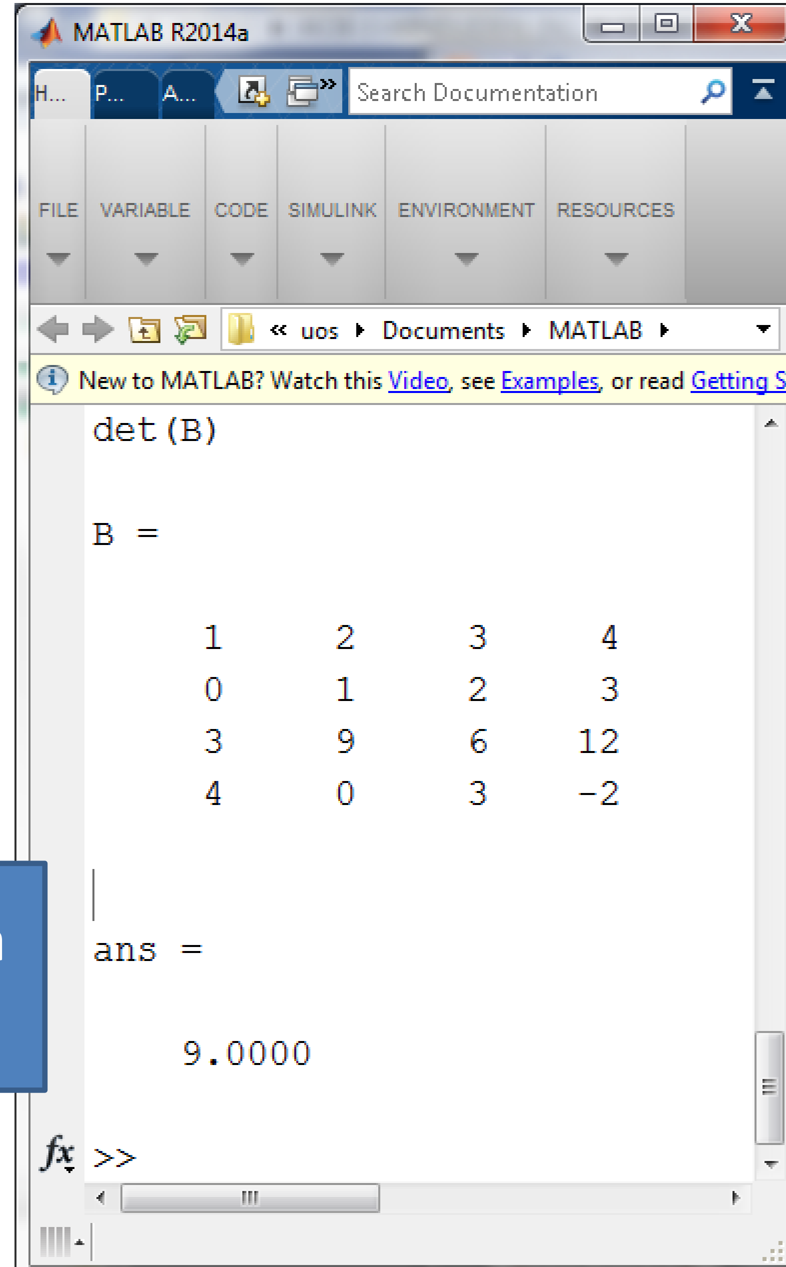
$$|A| = -6\begin{vmatrix} 0 & 3 \\ 7 & 8 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 3 \\ 4 & 8 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 4 & 7 \end{vmatrix}$$

$$|B| = -6\lambda\begin{vmatrix} 0 & 3 \\ 7 & 8 \end{vmatrix} + (-1)\lambda\begin{vmatrix} 1 & 3 \\ 4 & 8 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 4 & 7 \end{vmatrix}$$

Cofactors unaffected when expand along row 2

MATLAB example





Scale 3rd row by a factor of 3

SCALING EVERY ELEMENT BY K RESULTS IN A SCALING OF THE DETERMINANT BY K^N.

Scaling every element by λ results in a scaling of the determinant by λ^n.

First illustrate with a 2x2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad |A| = ad - bc = 0$$

$$B = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}; \quad |B| = \lambda^2 (ad - bc)$$

Scaling every element by λ results in a scaling of the determinant by λ^n.

Consider now a 3x3 example

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad B=\lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = -a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{12} & a_{13} \end{vmatrix} + a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{11} & a_{13} \end{vmatrix} - a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} = 0$$

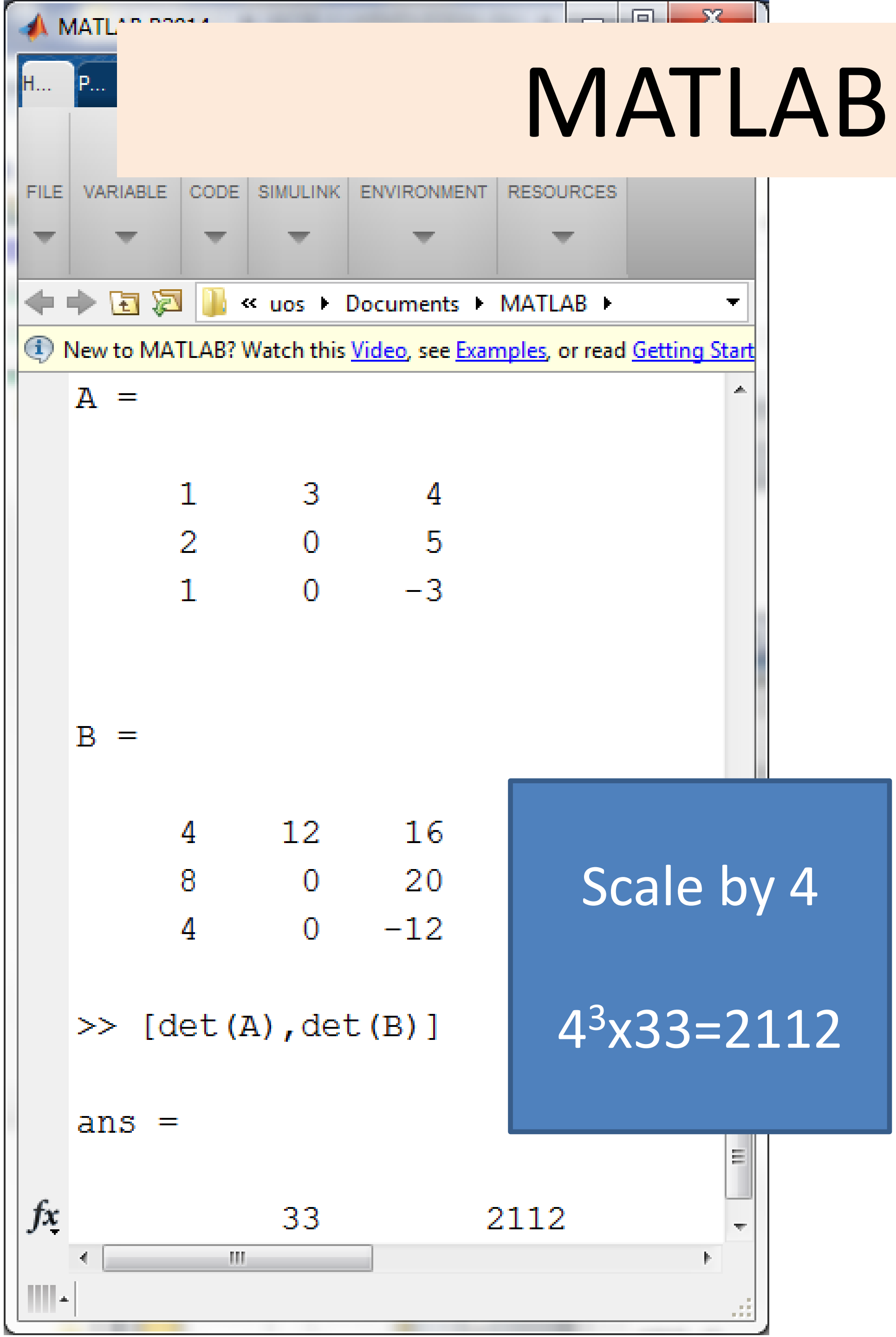
$$|B| = -\lambda a_{31}\begin{vmatrix} \lambda a_{12} & \lambda a_{13} \\ \lambda a_{22} & \lambda a_{23} \end{vmatrix} + \lambda a_{32}\begin{vmatrix} \lambda a_{11} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{23} \end{vmatrix} - \lambda a_{33}\begin{vmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{vmatrix} = \lambda^3|A|$$

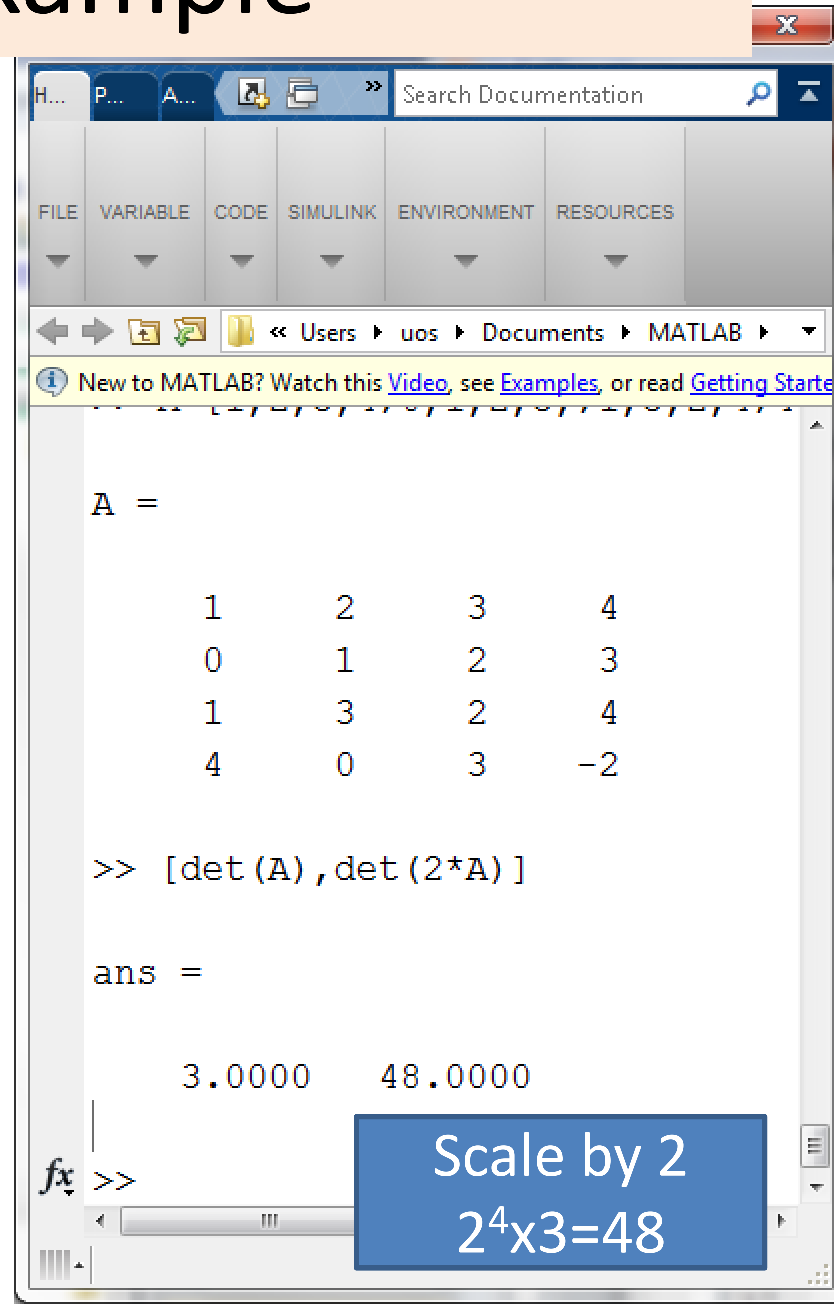
Scaling every element by λ results in a scaling of the determinant by λ^n.

For a 4x4 matrix and higher dimensions the general results should now be obvious and could be proved by recursion.

$$\left\{\begin{array}{c} |A| = \sum_{j=1}^{4} a_{ij} A_{ij} \\ b_{ij} = \lambda a_{ij} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} |B| = \sum_{j=1}^{4} b_{ij} B_{ij} \\ B_{ij} = \lambda^{n-1} A_{ij} \\ |B| = \lambda^{n} \sum_{j=1}^{4} a_{ij} A_{ij} \end{array}\right\}$$

MATLAB example





Numerical example

Given the known determinant, find the unknown determinant.

$$A = \begin{bmatrix} 1 & 6 & 5 \\ 2 & -1 & 0 \\ 1 & 4 & 2 \end{bmatrix}; \quad |A| = 19;$$

$$B = \begin{bmatrix} 1 & 6 & 5 \\ -6 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}; \quad |B| =$$

Numerical example

Given the known determinant, find the unknown determinant.

$$B = \begin{bmatrix} -1 & 4 & 5 & 0 \\ 3 & -12 & -15 & 1 \\ 5 & 18 & 4 & 7 \\ 104 & 56 & 13 & 2 \end{bmatrix}; \quad |B| = -6566$$

$$F = \begin{bmatrix} -1 & 4 & 20 & 0 \\ 3 & -12 & -60 & 1 \\ 5 & 18 & 16 & 7 \\ 104 & 56 & 52 & 2 \end{bmatrix}; \quad |F| = -$$ [illegible] 4

Numerical example

Given the known determinant, find the unknown determinant.

$$A = \begin{bmatrix} 1 & 6 & 5 \\ 2 & -1 & 0 \\ 1 & 4 & 2 \end{bmatrix}; \quad |A| = 19;$$

$$B = \begin{bmatrix} 2 & 12 & 10 \\ 4 & -2 & 0 \\ 2 & 8 & 4 \end{bmatrix}; \quad |B| =$$

Summary

1. For upper or lower triangular and diagonal matrices, the determinant is the product of the diagonal elements.
2. If a matrix has an entire row or column of zeros, the determinant is zero.
3. Scaling any row by λ results in a scaling of the determinant by λ.
4. Multiplying an nxn matrix by a scalar λ modifies the determinant by λ^n.