The University Of Sheffield.

Matrices 13: determinant properties and rules continued

Anthony Rossiter

http://controleducation.group.shef.ac.uk/indexwebbook.html

http://www.shef.ac.uk/acse

Department of Automatic Control and Systems Engineering

Introduction

- Previous videos introduced the concepts of a determinant but it was clear that in general these would be rather tedious to compute.
- This video introduces rules and shortcuts which allow much easier computation.
- For a matrix with coefficients a_{ij} and cofactors A_{ij}, determinant is defined from expansion along any row or column, that is:

$$\sum_{j=1}^{n} a_{ij} A_{ij}$$

$$\sum_{i=1}^{n} a_{ij} A_{ij}$$

Reminder of video 12

1. For upper or lower triangular and diagonal matrices, the determinant is the product of the diagonal elements.
2. If a matrix has an entire row or column of zeros, the determinant is zero.
3. Scaling any row by λ results in a scaling of the determinant by λ.
4. Multiplying an nxn matrix by a scalar λ modifies the determinant by λ^n.

In this video we develop properties which identify when a determinant might be zero thus saving unnecessary computation.

IF 2 ROWS/COLUMNS ARE IDENTICAL THEN THE DETERMINANT IS ZERO

If two rows are the same, the (2x2) determinant is zero.

We begin with a 2x2 example.

$$A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}; \quad |A| = ab - ba = 0$$

In this case the result falls out directly from the definition.

It is clear the same result follows if two columns are the same.

If two rows are the same, the (3x3) determinant is zero.

Without loss of generality, make row 2 equal to row 1.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad cof(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Use the determinant definition along row 3.

$$|A| = a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{12} & a_{13} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{11} & a_{13} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} = 0$$

Exploiting results from 2x2 determinants

If two rows are the same, the (3x3) determinant is zero.

Here we give an alternative proof for a 3x3 example.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad cof(A)=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ -A_{11} & -A_{12} & -A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

In this case the cofactors for row 2 must match those for row 1, but with opposite signs.

Use the determinant definition along row 1 and row 2.

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{11}(-A_{11}) + a_{12}(-A_{12}) + a_{13}(-A_{13})$$

$$|A| = -|A| \Rightarrow |A| = 0$$

Clearly |A| must be ZERO!!

If two rows are the same, the determinant of a 4x4 is zero.

This follows directly from the result for a 3x3.

Without loss of generality we will illustrate with row 4.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

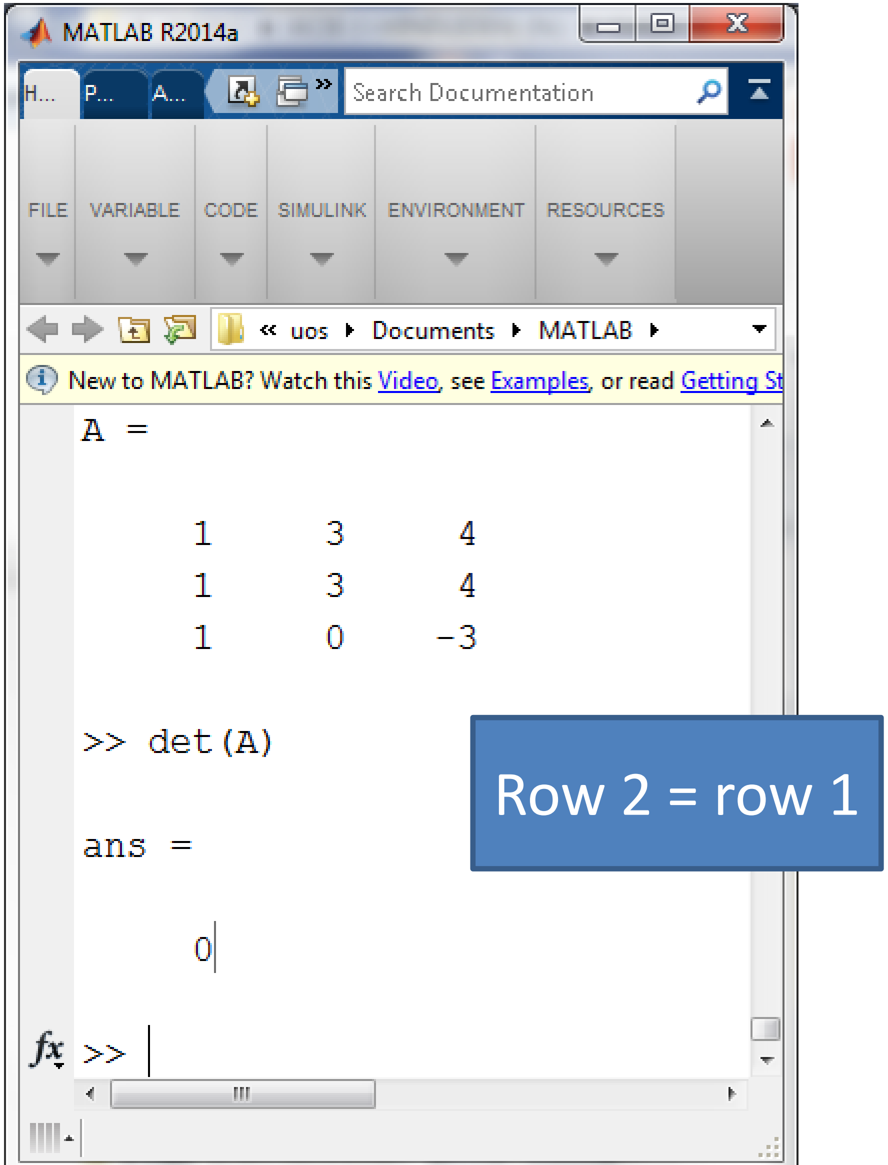
It is clear that every cofactor for the 4^{th} row is a made up of a 3x3 determinant, where the 3x3 determinant has a common rows. Hence use result of previous slide.

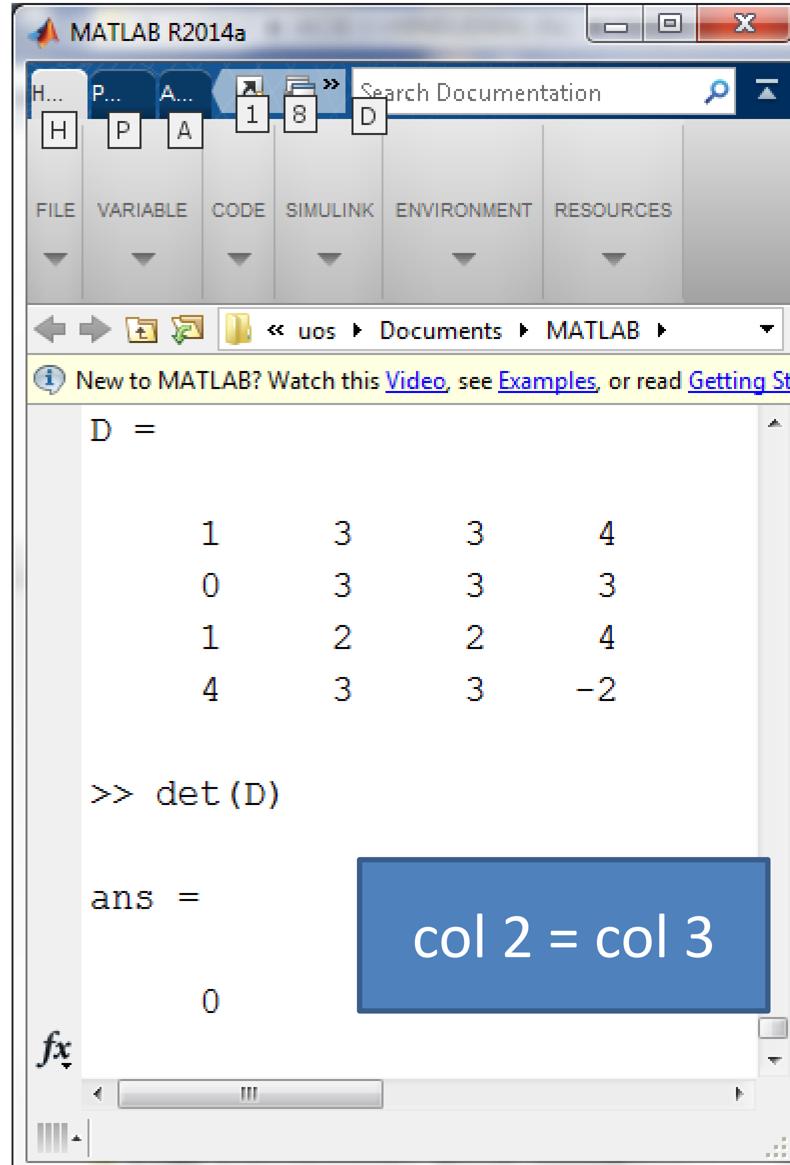
$$|A| = a_{41}A_{41} + a_{42}A_{42} + a_{43}A_{43} + a_{44}A_{44}$$

$$A_{41} = A_{42} = A_{43} = A_{44} = 0$$

Extension to common columns etc is obvious.

MATLAB example





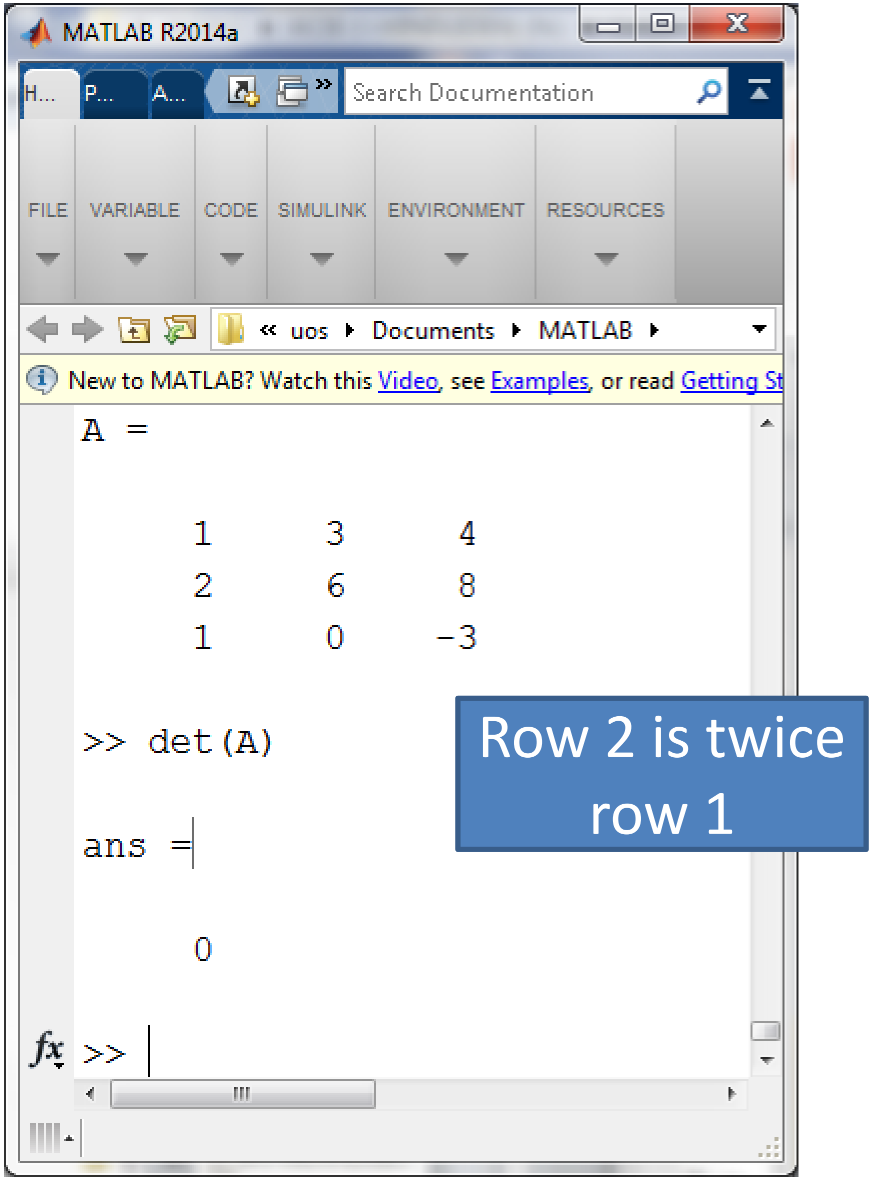
Corollary

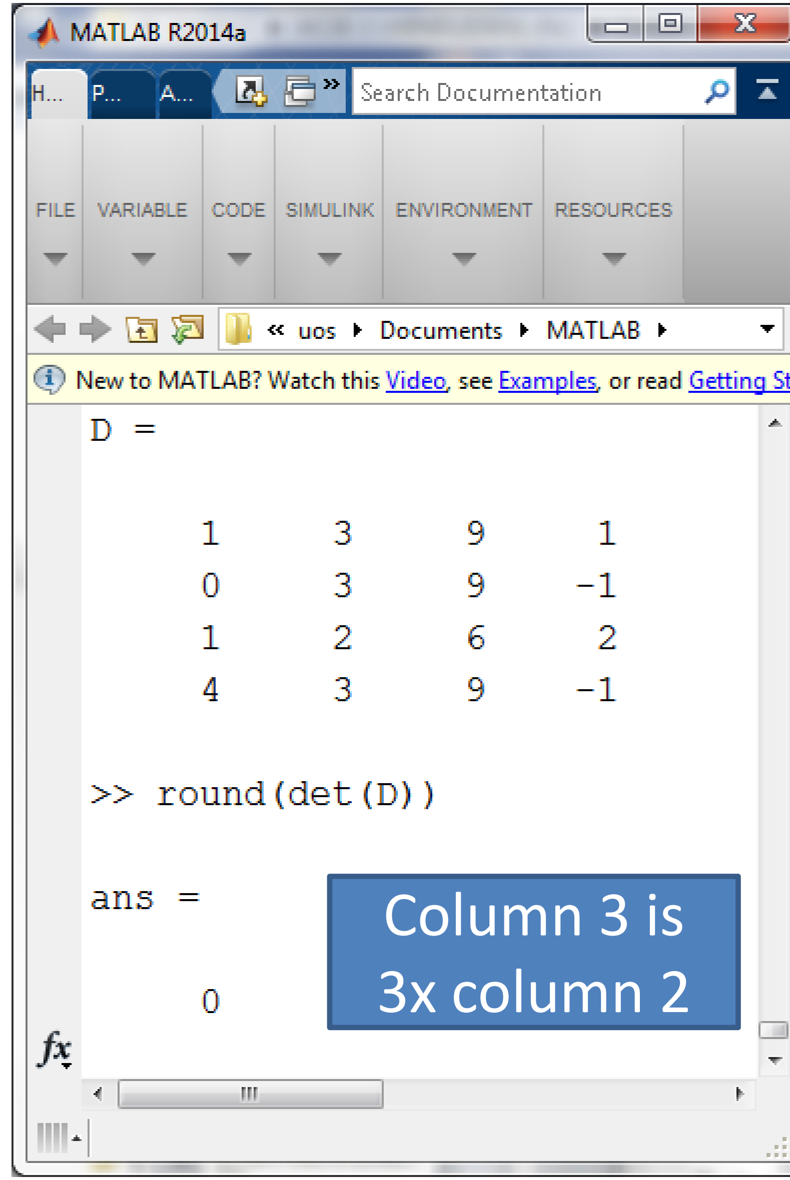
If a row (or column) is a multiple of another row (or column) then the determinant is zero.

This follows directly from the rule that:

1. **Scaling any row by λ results in a scaling of the determinant by λ.**
2. **Hence one could choose a λ to scale the rows to be exactly equal which gives a determinant of ZERO; hence the original determinant must have been zero.**

MATLAB example





ADDING A MULTIPLE OF ANY ROW TO ANOTHER ROW DOES NOT CHANGE THE DETERMINANT.

Adding a multiple of any row to another row does not change the determinant.

This results builds on the two earlier results.

1. If two rows are the same, the determinant is zero.
2. Scaling any row by λ results in a scaling of the determinant by λ.

This video demonstrates the result for common rows. Extension to common columns is obvious/equivalent.

Define the determinant after adding a row to another row.

Extension to common columns etc is obvious.

Without loss of generality we will illustrate with adding row 1 to row 2.

$$A=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}; \quad B=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1}+a_{1,1} & a_{2,2}+a_{1,2} & a_{2,3}+a_{1,3} & a_{2,4}+a_{1,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24}$$

$$|B| = (a_{21}+a_{11})A_{21} + (a_{22}+a_{12})A_{22} + (a_{23}+a_{13})A_{23} + (a_{24}+a_{14})A_{24}$$

Define the determinant after adding a row to another row.

Rearrange the determinant calculation back to an underlying matrix.

$$|B| = |A| + a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} + a_{14}A_{24} = |A| + |C|$$

$$|C| = a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} + a_{14}A_{24}$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}; \quad C = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

Clearly det(C)=0 as it has 2 common rows.

MATLAB examples

```
A =

     1     3     4
     1     2     8
     1     0    -3

>> B=A;B(2,:)=B(1,:)+B(2,:)

B =

     1     3     4
     2     5    12
     1     0    -3

>> [det(A),det(B)]

ans =

    19    19
```

Add row 1 to row 2

```
A =

     1     2     3     4
     0     1     2     3
     1     3     2     4
     4     0     3    -2

>> B=A;B(:,3)=B(:,3)+B(:,4)

B =

     1     2     7     4
     0     1     5     3
     1     3     6     4
     4     0     1    -2

>> [det(A),det(B)]

ans =

    3.0000    3.0000
```

Add col 4 to col 3

Define the determinant after adding a multiple of a row to another row.

Extension to common columns etc is obvious.

Without loss of generality we will illustrate by adding **λ x** row 1 to row 2.

$$A=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}; \quad B=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1}+\lambda a_{1,1} & a_{2,2}+\lambda a_{1,2} & a_{2,3}+\lambda a_{1,3} & a_{2,4}+\lambda a_{1,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24}$$

$$|B| = (a_{21}+\lambda a_{11})A_{21} + (a_{22}+\lambda a_{12})A_{22} + (a_{23}+\lambda a_{13})A_{23} + (a_{24}+\lambda a_{14})A_{24}$$

MATLAB examples

```
A =

     1     3     4
     1     2     8
     1     0    -3

>> B=A;B(3,:)=B(3,:)+0.4*B(2,:)

B =

    1.0000    3.0000    4.0000
    1.0000    2.0000    8.0000
    1.4000    0.8000    0.2000

>> [det(A),det(B)]

ans =

   19.0000   19.0000

>>
```

Add 0.4 row 2 to row 3

```
A =

     1     2     3     4
     0     1     2     3
     1     3     2     4
     4     0     3    -2

>> B=A;B(:,1)=B(:,1)-0.5*B(:,3)

B =

   -0.5000    2.0000    3.0000    4.0000
   -1.0000    1.0000    2.0000    3.0000
         0    3.0000    2.0000    4.0000
    2.5000         0    3.0000   -2.0000

>> [det(A),det(B)]

ans =

    3.0000    3.0000
```

Subtract 0.5 col 3 from col 1

Find the determinant (using properties)

$$B = \begin{bmatrix} -1 & 4 & 5 & 0 \\ 3 & -12 & -15 & 0 \\ 5 & 18 & 3 & 7 \\ 104 & 56 & 13 & 2 \end{bmatrix}$$

Find the determinant (using properties)

$$\begin{bmatrix} -1 & 2 & -10 & 4 \\ 3 & 1 & -5 & 2 \\ 5 & -6 & 30 & -13 \\ 24 & -4 & 21 & -9 \end{bmatrix}$$

Summary of rules

1. For upper or lower triangular and diagonal matrices, the determinant is the product of the diagonal elements.
2. If a matrix has an entire row or column of zeros, the determinant is zero.
3. Scaling any row by λ results in a scaling of the determinant by λ.
4. Multiplying an nxn matrix by a scalar λ modifies the determinant by λ^n.
5. Adding a multiple of any row to another row does not change the determinant.
6. If two rows (or two columns) are equal then the determinant is zero.
7. If multiple of a row (col) is equal to another row (col) then the determinant is zero.