Matrices 14: determinant rules continued

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http://controleducation.group.shef.ac.uk/indexwebbook.html

http://www.shef.ac.uk/acse

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Introduction

- Previous video introduced rules and shortcuts which allow much easier computation.
- This video introduces two last rules.

Reminder of rules in videos 12-13

1. For upper or lower triangular and diagonal matrices, the determinant is the product of the diagonal elements.
2. If a matrix has an entire row or column of zeros, the determinant is zero.
3. Scaling any row by λ results in a scaling of the determinant by λ.
4. Multiplying an nxn matrix by a scalar λ modifies the determinant by λ^n.
5. Adding a multiple of any row to another row does not change the determinant.
6. If two rows (or two columns) are equal then the determinant is zero.
7. If multiple of a row (col) is equal to another row (col) then the determinant is zero.

INTERCHANGING ROWS (OR COLUMNS) CHANGES THE SIGN OF THE DETERMINANT.

Interchange the columns of a 2x2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad |A| = ab - cd$$

$$B = \begin{bmatrix} b & a \\ d & c \end{bmatrix}; \quad |B| = bc - ad$$

Clearly swapping the columns has changed the sign of the determinant.

Interchange the columns of a 3x3 matrix

Without loss of generality, swap col 2 and col 1.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad B = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$$

Use the determinant definition along row 3.

$$|A| = a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

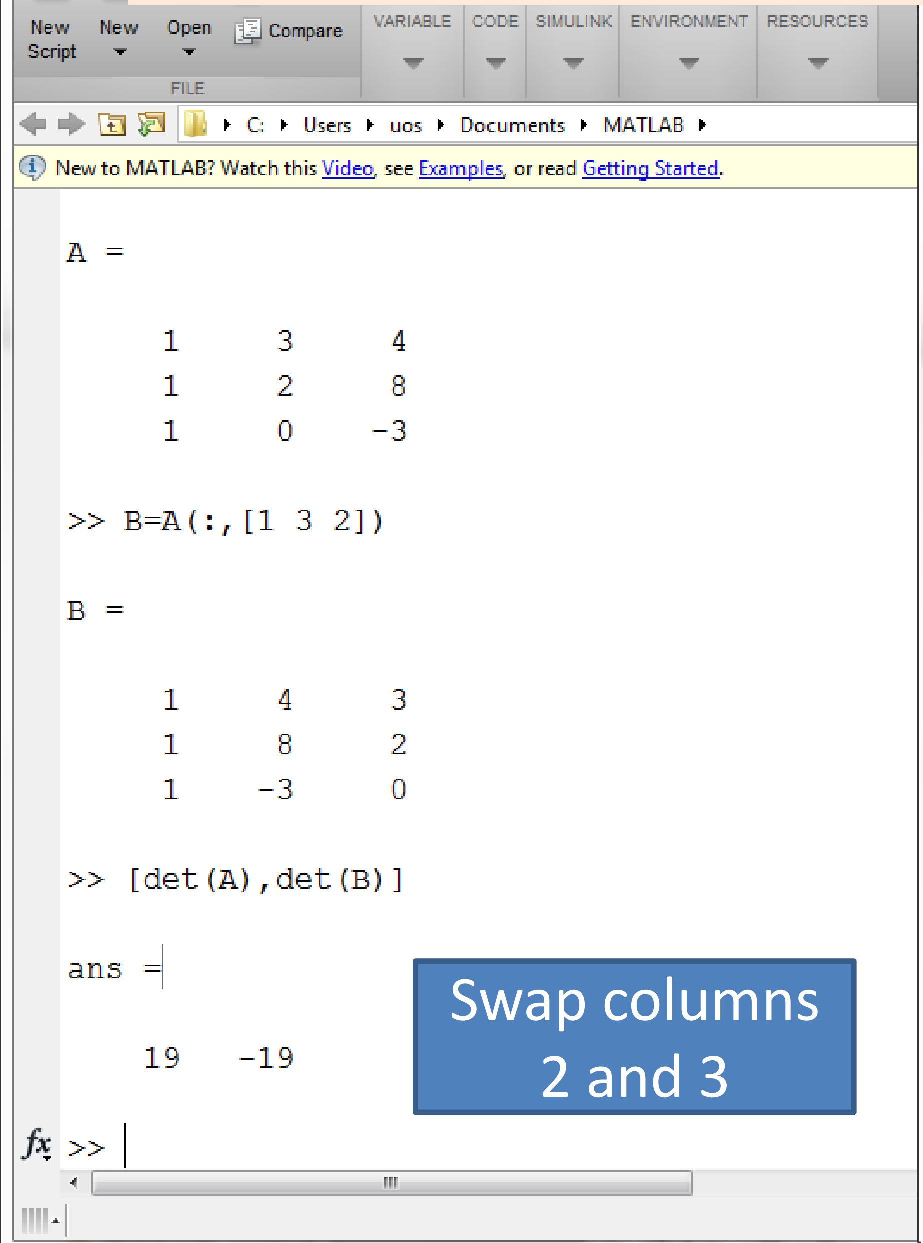
$$|B| = a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} - a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix}$$

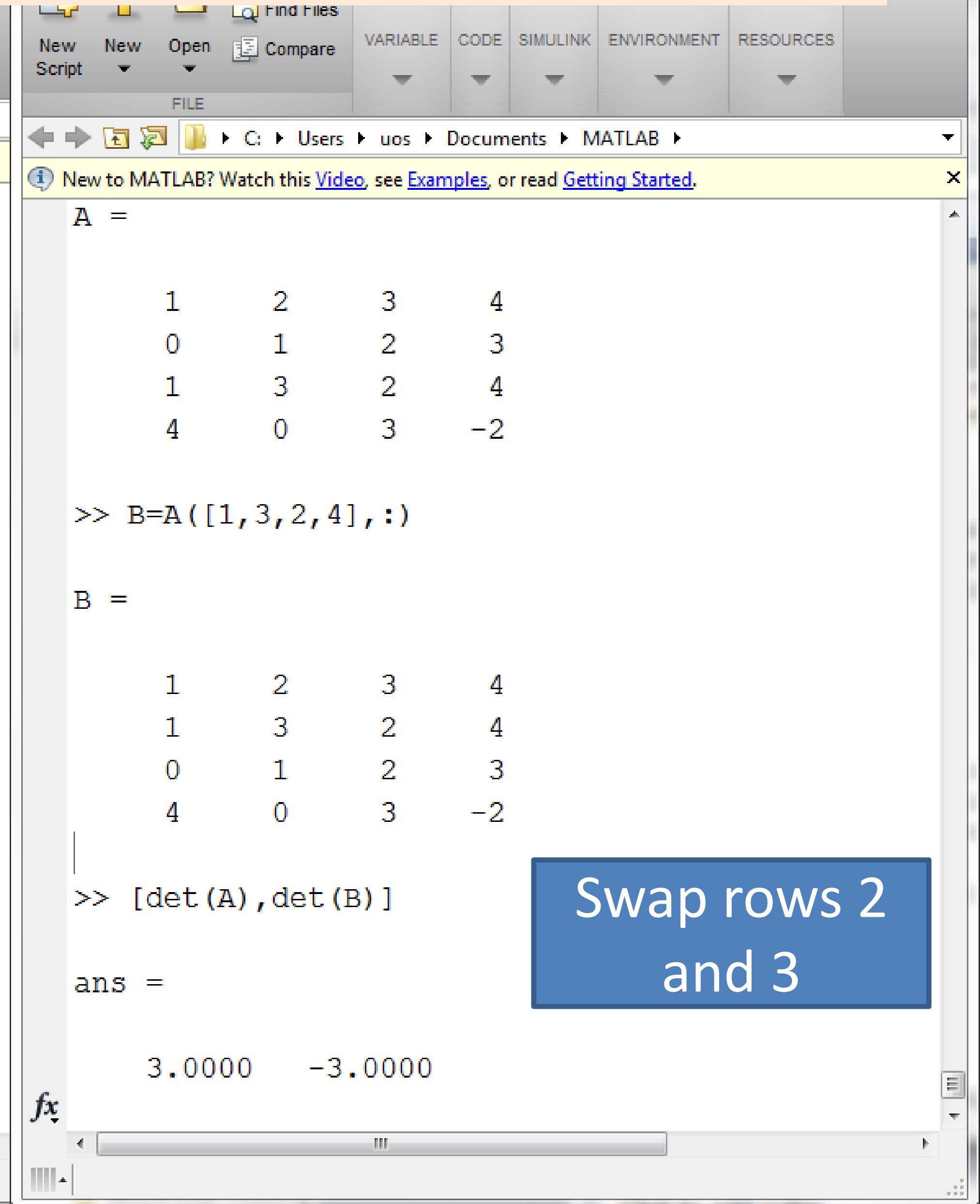
If two columns/rows are swapped, the determinant changes sign.

This follows directly from the insights with the 2x2 and 3x3 examples due to the ‘sign matrix’ meaning that cofactor computations appear with the opposite sign to previously.

We will not give 4x4 and higher dimension examples in detail as this is straightforward for viewers to establish.

MATLAB example





$$A = \begin{bmatrix} -4 & 2 & -10 & 8 \\ 3 & 0 & 3 & 2 \\ 2 & 0 & 0 & -12 \\ -6 & 0 & 0 & -9 \end{bmatrix} \quad |A| = -\left\| \begin{bmatrix} 2 & -4 & -10 & 8 \\ 0 & 3 & 3 & 2 \\ 0 & 2 & 0 & -12 \\ 0 & -6 & 0 & -9 \end{bmatrix} \right\|$$

$$|A| = +\left\| \begin{bmatrix} 2 & -10 & -4 & 8 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 2 & -12 \\ 0 & 0 & -6 & -9 \end{bmatrix} \right\|$$

THE DETERMINANT OF A PRODUCT IS THE PRODUCT OF THE DETERMINANTS

The determinant of a product is the product of the determinants

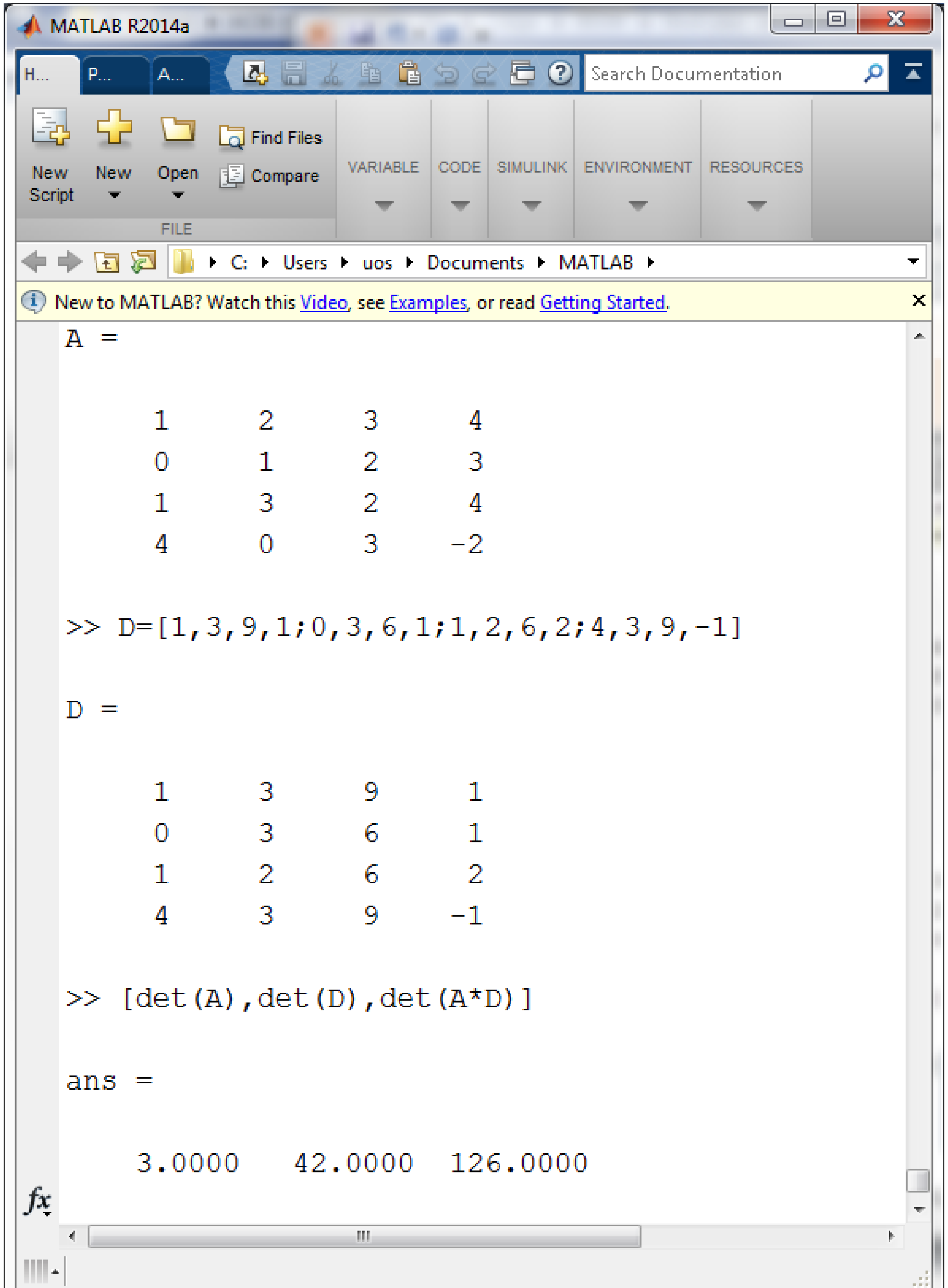
This rule is demonstrated without proof.

In summary it means that:

$$|AB| = |A||B|; \qquad |ABC| = |A||B||C|$$

This is useful because it means one can avoid multiplying out matrices if only the determinant is required.

MATLAB example



Summary of rules

1. For upper or lower triangular and diagonal matrices, the determinant is the product of the diagonal elements.
2. If a matrix has an entire row or column of zeros, the determinant is zero.
3. Scaling any row by λ results in a scaling of the determinant by λ.
4. Multiplying an nxn matrix by a scalar λ modifies the determinant by λ^n.
5. Adding a multiple of any row/col to another row/col does not change the determinant.
6. If two rows (columns) are equal then the determinant is zero.
7. If multiple of a row (col) is equal to another row (col) then the determinant is zero.
8. Interchanging rows (or cols) changes the sign of the determinant.
9. The determinant of a product is the product of the determinants.

Objective

A judicious use of the rules allows difficult determinant computations to be reduced to simple ones.

This is a core skill, not just for pen and paper computations, but also for generating efficient computer code.