The University Of Sheffield.

Matrices 2: special matrices and equality

Anthony Rossiter

http://controleducation.group.shef.ac.uk/indexwebbook.html

http://www.shef.ac.uk/acse

Department of Automatic Control
and Systems Engineering

Introduction

- The previous video introduced definitions of matrices and basic notation.
- This video defines some ‘special’ matrices that come up a lot in engineering problem solving.
- It is also considers the concept of equality.

Reminder

A matrix in essence is a table of numbers.

1. The dimensions of a matrix are the number of rows and the number of columns.
2. Notation for defining elements of a matrix assumes the top left is row 1 and column 1.

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \end{bmatrix}$$

A is *3 by 5*
because it has
3 rows and 5 columns.

Row matrix

A row matrix is a matrix which has a single row.

$$A = \begin{bmatrix} 2 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} -3 & 0 & 9 \end{bmatrix}; \quad C = \underbrace{\begin{bmatrix} 4 & 7 & \cdots & 5 \end{bmatrix}}_{n \text{ elements}}$$

Here:

- A is 1 by 2
- B is 1 by 3
- C is 1 by n

Some times you will here row matrices referred to as **row vectors** or even just **vectors**.

Column matrix

A column matrix is a matrix which has a single column.

Here:

- A is 2 by 1
- B is 3 by 1
- C is n by 1

$$A = \begin{bmatrix} -3 \\ 0 \end{bmatrix}; \quad B = \begin{bmatrix} 2 \\ -3 \\ 21 \end{bmatrix}; \quad C = \begin{bmatrix} 5 \\ -2 \\ \vdots \\ 6 \end{bmatrix}$$

Some times you will here column matrices referred to as

Column vectors

or even just

vectors.

Notation for vectors

As a vector has only one dimension greater than one, it is common to denote elements using just one subscript and also to use lower case letters to emphasis this is a vector.

$$a = \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$c = \begin{bmatrix} 5 \\ -2 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

Notation for row vectors

The default for a vector is the column form.

To avoid confusion, row vectors are usually denoted explicitly as ‘row’ vectors or using a transpose operation (discussed later).

Square matrices

Matrices have 2 dimensions, n**umber of rows and number of columns**.

A matrix is square if the two dimensions are the same.

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 0 & 0 \\ -5 & 2 & -4 \end{bmatrix}$$

3 rows and 3 columns.

$$A = \begin{bmatrix} -1 & 4 & 5 & 0 \\ 3 & -12 & -15 & 0 \\ 5 & 18 & 3 & 7 \\ 104 & 56 & 13 & 2 \end{bmatrix}$$

4 rows and 4 columns.

Identity matrix – usually called I

The identity matrix is ALWAYS SQUARE.

It comprises zero elements except on the principal diagonal whose elements are one.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $\{I_{ij}=0, i \neq j\}, \{I_{ij}=1, i=j\}$

Standard basis set

The columns (or equivalently rows) of the identity matrix form the standard basis set.

Its common, but not essential, to denote these as e_i.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

It is important to know the underlying space dimension to define e_i, so for example:

e_6^T=[0 0 0 0 0 1] or [0 0 0 0 0 1 0 0 0] or ….

Matrix equality

We need a mechanism for determining when two matrices A and B are equal. They must have:

- the **same dimensions.**
- **identical elements** in the corresponding positions.

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix}; \quad B = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix}$$

$$\{A = B\} \quad \Rightarrow \quad \{A_{i,j} = B_{i,j}, \forall i, j\}$$

Matrix transpose operation

Transposition is a very important operation so students must be clear on how it is **DEFINED**.

In simple terms, transposition swaps the indices so swaps **elements** in the corresponding positions.

$$\{A = B^T\} \Rightarrow \{A_{i,j} = B_{j,i}, \forall i, j\}$$

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 0 & 0 \\ -5 & 2 & -4 \end{bmatrix} \Rightarrow A = B^T = \begin{bmatrix} 1 & 5 & -5 \\ 0 & 0 & 2 \\ 3 & 0 & -4 \end{bmatrix}$$

i^{th} column becomes i^{th} row (or vice versa)

Matrix transposition examples

Transposition swaps the matrix dimensions.

If **B is n by r** then **B^T is r by n**.

This is obvious from the definition.

$$\{A = B^T\} \Rightarrow \{A_{i,j} = B_{j,i}, \forall i, j\}$$

$$H = \begin{bmatrix} 2 \\ -3 \\ 21 \end{bmatrix} \Rightarrow H^T = \begin{bmatrix} 2 & -3 & 21 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 2 & 4 & 4 \\ 1 & 0 & 6 & 2 & -1 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 6 \\ 4 & 2 \\ 4 & -1 \end{bmatrix}$$

Examples of Matrix transpose

$$A_{i,j} = A^T_{j,i}$$

$$A = \begin{bmatrix} 2 & 3 \\ -6 & 4 \\ 3 & 4 \end{bmatrix}; \qquad A^T = \begin{bmatrix} 2 & -6 & 3 \\ 3 & 4 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 21 & 5 & -4 & 0 & 4 & 0 \end{bmatrix}; \quad C^T = \begin{bmatrix} 21 \\ 5 \\ -4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

Alternative insight.

The jth column of A is the jth row of A^T.

Matrix transposition examples

What is the transpose of the following?

$$\left\{B = A^{T}\right\} \Rightarrow \left\{A_{i,j} = B_{j,i}, \forall i, j\right\}$$

$$A = \begin{bmatrix} -1 & 4 & 5 & 0 \\ 3 & -12 & -15 & 0 \\ 5 & 18 & 3 & 7 \\ 104 & 56 & 13 & 2 \end{bmatrix} \Rightarrow \quad A^{T} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Alternative insight

The jth column of A is the jth row of A^T.

Matrix transpose examples

Find the transpose of the following:

$$D = \begin{bmatrix} 8 & -3 & 2 \\ 6 & 5 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 3 & -3 & 2 & 4 \\ 4 & 5 & 1 & 5 \\ 2 & 4 & 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Which is the correct transpose of G?

$$G=\begin{bmatrix}2 & 2\\ 2 & -2\\ 2 & -2\end{bmatrix};\quad A=\begin{bmatrix}2 & -2 & -2\\ 2 & 2 & 2\end{bmatrix};\quad B=\begin{bmatrix}2 & 2 & 2\\ -2 & -2 & 2\end{bmatrix}$$

$$C=\begin{bmatrix}2 & 2 & -2\\ 2 & 2 & -2\end{bmatrix};\quad D=\begin{bmatrix}2 & 2 & 2\\ 2 & -2 & -2\end{bmatrix}$$

Symmetric matrices

A matrix is symmetric if $A=A^T$.

This means $\{A = A^T\} \Rightarrow \{A_{i,j} = A_{j,i}, \forall i, j\}$

IMPLICITLY, a matrix can only be symmetric if it is **also square** (as number of rows/cols swap).

$$G = \begin{bmatrix} 10 & -2 \\ -2 & 6 \end{bmatrix}; \quad G^T = \begin{bmatrix} 10 & -2 \\ -2 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & -4 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & -4 \end{bmatrix}$$

Summary

- Defined row and column matrices (also denoted as vectors).
- Defined square matrices and identity matrices.
- Defined the standard basis set vectors.
- Defined matrix equality.
- Defined the transposition operation.
- Defined symmetric matrices.