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Matrices 6: properties of multiplication

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http://controleducation.group.shef.ac.uk/indexwebbook.html

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Introduction

- The previous video multiplication of matrices.
- Here we look at some useful properties of matrices linked to the multiplication operation.

Generic formulae for matrix multiplication

The following formulae is used.

$$C = AB \quad \Rightarrow \quad C_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots A_{i,n}B_{n,j}$$

{i,j} element of the result uses the ith row of the left hand matrix and the jth column of the right hand matrix.

Result has row dimension of A and column dimension of B.

KEY REMINDER

Two matrices can only be multiplied if the column dimension of the left matrix matches the row dimension of the right matrix.

In general

$$AB \neq BA$$

In fact, it is possible that AB exists but BA does not!

Properties 1

Prove that both pre- and post-multiplication of a matrix by an identity matrix results in no change.

1. A typical identify matrix is:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Use the formulae:

$$C = AI \quad \Rightarrow \quad C_{i,j} = A_{i,1}I_{1,j} + A_{i,2}I_{2,j} + \cdots + A_{i,n}I_{n,j}$$

3. Noted that $I_{i,j}=0$ *if* $i \neq j$ and hence*:*

$$C_{i,j} = A_{i,j}I_{j,j} = A_{i,j}$$

A similar proof works for D=IA.

Properties 2

Prove that AB=0 does not imply that either A=0 or B=0. [Note the difference with scalars].

This can be demonstrated with a simple example based on vectors. If it applies for one example, it can clearly apply for many others.

$$\begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 0 & -3 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Vectors whose product is zero are orthogonal!

Properties 3

Prove that A *(BC) = (AB)*C, that is, the order of multiplication does not affect the result.

$$D = AB \quad \Rightarrow \quad D_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots$$

$$E = ABC \quad \Rightarrow \quad E_{i,j} = D_{i,1}C_{1,j} + D_{i,2}C_{2,\cdots}$$

$$E_{i,j} = (A_{i,1}B_{1,1} + A_{i,2}B_{2,1} + \cdots + A_{i,n}B_{n,1})C_{1,j} + \cdots$$

$$+ (A_{i,1}B_{1,m} + A_{i,2}B_{2,m} + \cdots + A_{i,n}B_{n,m})C_{m,j}$$

CLEARLY THE SAME FORMULAE

$$F = BC \quad \Rightarrow \quad F_{i,j} = B_{i,1}C_{1,j} + B_{i,2}C_{2,j} + \cdots + B_{i,m}C_{m,j}$$

$$E = ABC \quad \Rightarrow \quad E_{i,j} = A_{i,1}F_{1,j} + A_{i,2}F_{2,j} + \cdots + A_{i,n}F_{n,j}$$

$$E_{i,j} = A_{i,1}(B_{1,1}C_{1,j} + B_{1,2}C_{2,j} + \cdots + B_{1,m}C_{m,j}) + \cdots$$

$$+ A_{i,n}(B_{n,1}C_{1,j} + B_{n,2}C_{2,j} + \cdots + B_{n,m}C_{m,j})$$

Properties 4

A matrix can always be multiplied by its own transpose.

1. Matrix A has dimensions r by n
2. Matrix A^T has dimensions n by r.

3. $A*A^T$ works as the column dimension of A is n and the row dimension of A^T is n.
4. A^T*A works as the column dimension of A^T is r and the row dimension of A is r.

Properties 5 - commutativity

A matrices are said to be commutative if the following property holds:

$$AB = BA$$

The conditions for this happening are quite restrictive and hence it is quite rare in general.

It should be obvious that commutative matrices must be square (otherwise the dimensions of AB,BA would not match).

Summary

Introduced some key properties linked to matrix multiplication.

1. Impact of an identity matrix.
2. Possibility of the product of two non-zero matrices being zero.
3. Insignificance of the order of multiplication.
4. Concept of commutativity.