Modelling and control summaries

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Modelling principles and analogies 9:

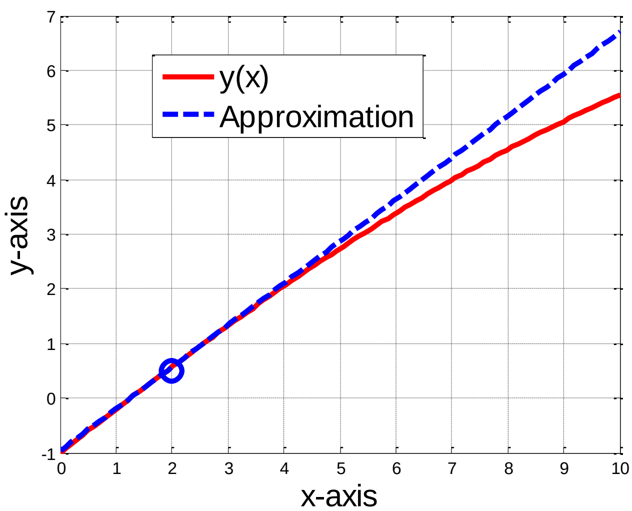
Linearising non-linear systems

It is not possible to use superposition with non-linear systems and as a consequence, conventional analysis techniques are invalid. It is convenient therefore to ask whether a nonlinear system can be approximated, locally, by a linear system.

Illustration of linearisation of a simple curve.

$$y(x) = \sin(0.2x) + 0.5x - e^{-0.1x}$$

Clearly, a straight line approximation is valid for a small domain – the figure demonstrates a straight line around the point x=2.



Techniques for approximation and linearization

The most common technique for linearising around a point x_s is Taylor series, hence:

$$f(x) - f(x_s) = \left(\frac{df(x)}{dx}\right)_{x=x_s}\left(\frac{x-x_s}{1!}\right) + \left(\frac{d^2 f(x)}{dx^2}\right)_{x=x_s}\left(\frac{x-x_s}{2!}\right)^2 + \cdots$$

Neglecting higher order terms, and writing in terms of deviation variables, this reduces to a linear system.

$$f(x) - f(x_s) = f'(x_s)(x - x_s); \quad f' = \frac{df}{dx}; \quad y = f(x)$$

$$\delta y = f'(x_s)\delta x;$$
$$\delta y = y - y_s, \quad \delta x = x - x_s$$

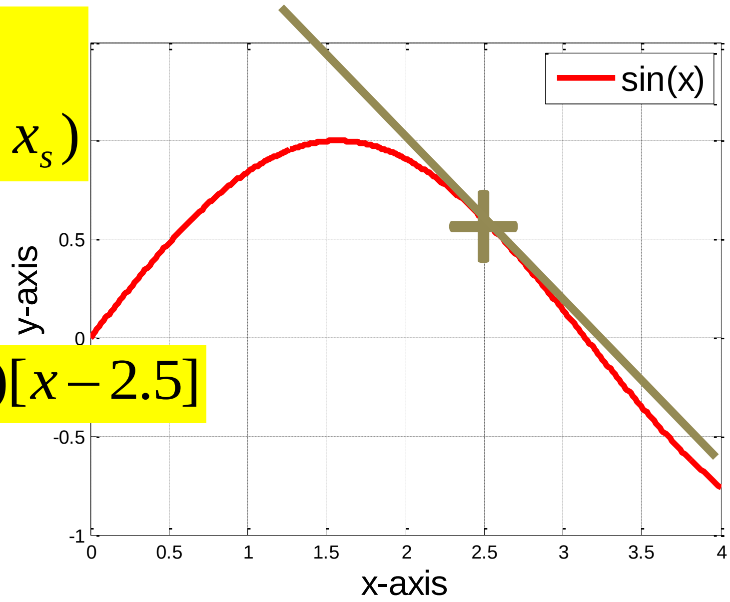
EXAMPLE 1: Linearise sine(x) around x=2.5

$$f'(x_s) = \cos(x_s)$$

$$y - y_s = [\cos(x_s)](x - x_s)$$

If wanted the straight line is given as:

$$y - \sin(2.5) = \cos(2.5)[x - 2.5]$$



EXAMPLE 2: Linearise the flow through a restriction. Represent the flow equation around a depth of h=1.2 with a 1st order Taylor series.

$$F(t) = \beta\sqrt{h(t)}$$

Step 1: Choose the point (here h=1.2)	$F_s = \beta\sqrt{h_s} = \beta\sqrt{1.2}$
Step 2: Find the derivative (at h=1.2)	$\frac{dF}{dh} = \frac{1}{2}\frac{\beta}{\sqrt{h}} \Rightarrow \frac{1}{2}\frac{\beta}{\sqrt{1.2}}$
Step 3: Substitute into the equation $\delta F = F'(h_s)\delta h;$	$\delta F = (\frac{1}{2}\frac{\beta}{\sqrt{1.2}})\delta h;$ $\delta h = h - 1.2$ $\delta F = F - \beta\sqrt{1.2}$

Remark

In general, the input/output relationship must be proportional for a system to be linear, that is:

output = k * input

A first order Taylor series achieves this form if the input and output are written as deviation variables.