Assume a standard time constant form for a 1st order model with constant coefficients, output $y(t)$ and input $u(t)$.

$$ T \frac{dy(t)}{dt} + y(t) = K \ u(t) $$

**Creating a time constant form**

$$ \left\{ a \frac{dy(t)}{dt} + by(t) = c \ u(t) \right\} \quad \equiv \quad \left\{ a \frac{dy(t)}{b} + y(t) = \frac{c}{b} \ u(t) \right\}; \quad T = \frac{a}{b}; \quad K = \frac{c}{b} $$

**Response with $u(t)=0$**

To confirm this, substitute the solution for $x(t)$ into the original differential equation and verify that it satisfies both the equation and the initial condition that $x(t)=x(0)$ when $t=0$.

$$ T \frac{dx}{dt} + x = 0 \quad \Rightarrow \quad x(t) = x(0) e^{-\frac{t}{T}} $$

**Verify solution using Laplace transforms**

$$ T \frac{dx}{dt} + x = 0 \quad \Rightarrow \quad \begin{cases} T \frac{dx}{dt} + x = 0 \\ L[x(t)] = X(s) \\ L[\frac{dx}{dt}] = sX(s) - x(0) \end{cases} $$

$$ \Rightarrow \quad [sT + 1]X(s) - Tx(0) = 0 \quad \downarrow \quad X(s) = \frac{x(0)}{s+1/T} $$

**Response dependence on the time constant**

\[
\begin{align*}
x(t) &= x(0) e^{-\frac{t}{T}} \\
x(T) &= x(0) e^{-\frac{T}{T}} = x(0) e^{-1} \\
x(2T) &= x(0) e^{-\frac{2T}{T}} = x(0) e^{-2} \\
x(3T) &= x(0) e^{-\frac{3T}{T}} = x(0) e^{-3}
\end{align*}
\]

\[ e^{-1} = 0.37, \quad e^{-2} = 0.14 \]

\[ e^{-3} = 0.05, \quad e^{-4} = 0.02 \]

The rate of decay is explicitly linked to the time constant $T$. Note above how the time axis is presented in terms of $T$ for convenience.
Sketch the response for.

1. Find the time constant. Here $T=1.5$.
2. Calculate $x(T)$, $x(2T)$, $x(3T)$ and draw a smooth curve between them.

First order responses with no system input are characterised by simple observations.

1. Response begins at initial condition and finishes at zero.
2. Follows a simple exponential curve which is determined by the time constant $T$.
3. Use multiples of $T$ on the time axis, then the curve is invariant (100% at 0 sec, 37% at $T$ sec, 14% at $2T$ sec, 5% at $3T$ sec, etc.).

REMARK
A simple method for estimating the time constant ‘T’ of a 1st order model is to plot the response from a non-zero initial condition and determine how long it takes to get to 37% of the original value. This can be quite useful with real hardware as such responses are easy to produce and plot.