Modelling and control summaries

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1^{st} order responses 1: No input

Assume a standard time constant form for a 1^{st} order model with constant coefficients, output y(t) and input u(t).	$T\frac{dy(t)}{dt}+y(t)=K\,u(t)$
Creating a time constant form	$\left\{a\frac{dy(t)}{dt}+by(t)=c\,u(t)\right\} \equiv \left\{\frac{a}{b}\frac{dy(t)}{dt}+y(t)=\frac{c}{b}u(t)\right\}; \quad T=\frac{a}{b}; \quad K=\frac{c}{b}$

Response with u(t)=0

To confirm this, substitute the solution for *x(t)* into the original differential equation and verify that it satisfies both the equation and the initial condition that *x(t)=x(0)* when *t=0.*

$$T\frac{dx}{dt}+x=0 \quad \Rightarrow \quad x(t)=x(0)e^{-\frac{t}{T}}$$

Verify solution using Laplace transforms

$$\left.\begin{array}{l} T\frac{dx}{dt}+x=0 \\ L[x(t)]=X(s) \\ L[\frac{dx}{dt}]=sX(s)-x(0) \end{array}\right\} \Rightarrow \begin{array}{c} [sT+1]X(s)-Tx(0)=0 \\ \Downarrow \\ X(s)=\frac{x(0)}{s+1/T} \end{array}$$

Response dependence on the time constant

$$x(t)=x(0)e^{-\frac{t}{T}}$$

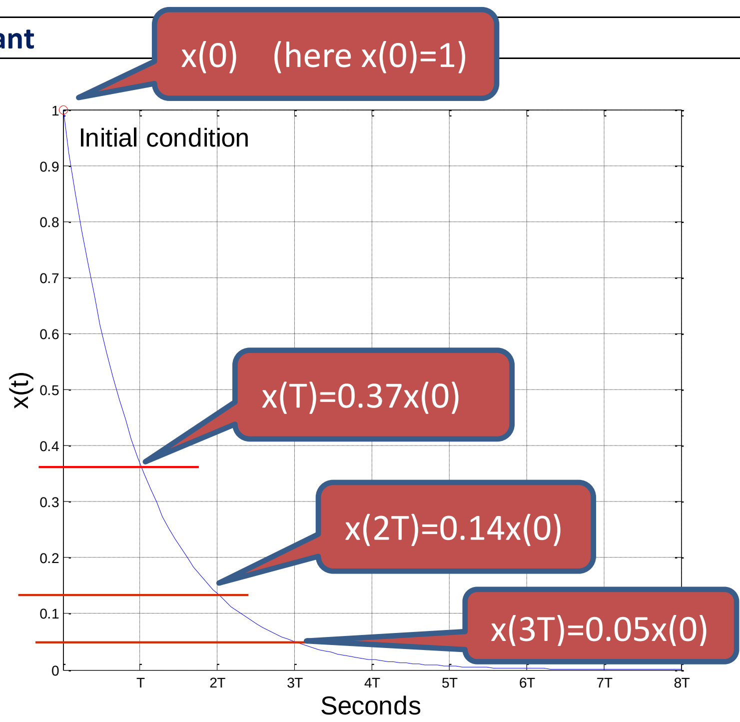
$$x(T)=x(0)e^{-\frac{T}{T}}=x(0)e^{-1}$$

$$x(2T)=x(0)e^{-\frac{2T}{T}}=x(0)e^{-2}$$

$$x(3T)=x(0)e^{-\frac{3T}{T}}=x(0)e^{-3}$$

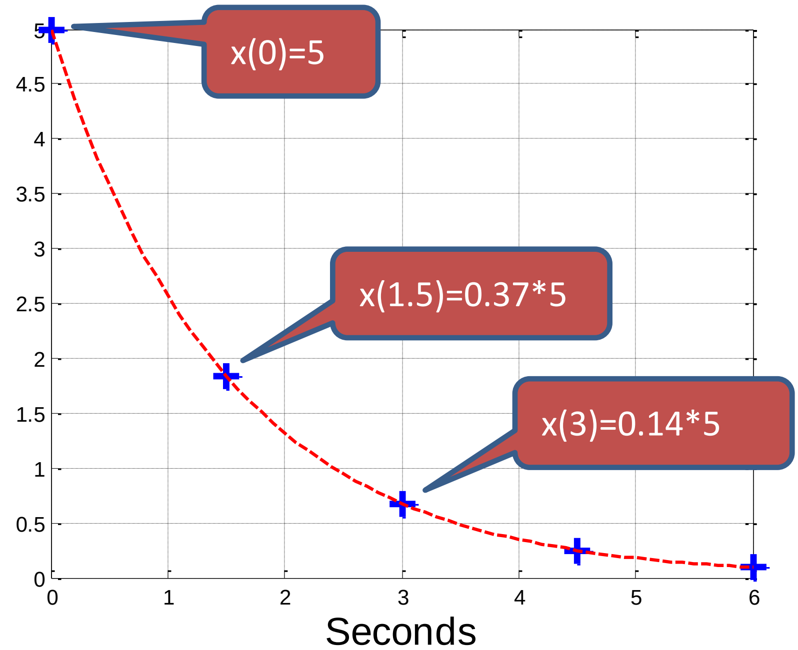
$$e^{-1}=0.37,\ e^{-2}=0.14$$

$$e^{-3}=0.05,\ e^{-4}=0.02$$



The rate of decay is explicitly linked to the time constant T. Note above how the time axis is presented in terms of T for convenience.

<table>
<tr><td>Sketch the response for.</td><td>$$\left\{6\frac{dx}{dt}+4x=0;\quad x(0)=5\right\}\quad\Rightarrow\quad 1.5\frac{dx}{dt}+x=0$$</td></tr>
<tr><td>1. Find the time constant. Here T=1.5.

2. Calculate x(T), x(2T), x(3T) and draw a smooth curve between them.</td><td>
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First order responses with no system input are characterised by simple observations.

1. Response begins at initial condition and finishes at zero.
2. Follows a simple exponential curve which is determined by the time constant T.
3. Use multiples of T on the time axis, then the curve is invariant (100% at 0 sec, 37% at T sec, 14% at 2T sec, 5% at 3T sec, etc.).

REMARK

A simple method for estimating the time constant 'T' of a 1st order model is to plot the response from a non-zero initial condition and determine how long it takes to get to 37% of the original value.

This can be quite useful with real hardware as such responses are easy to produce and plot.