Modelling and control summaries

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2nd order responses 1: over-damped systems

Assume a standard 2nd order model with constant coefficients, output y(t) and input u(t). What is the expected behaviour and how does this vary with a,b,c?	$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f$
For now the focus is on constant f, usually f=0 or f=1.	

An overdamped system is where the characteristic polynomial marked below has real roots

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f \quad\Rightarrow\quad ap^2+bp+c=0 \quad\Rightarrow\quad p=p_1,p_2$$

Real roots p_1,p_2 implies that $b^2>4ac$. This means that, for constant f, the solution takes the following form:	$x(t)=Ae^{p_1t}+Be^{p_2t}+\frac{f}{c}$

What remains is to determine the unknown coefficients A,B.

Response with zero initial conditions (x(0)=0, dx/dt(0)=0)

Substitute the known solution from above into the two initial conditions and this gives simultaneous equations in A and B.

$$\left\{\begin{array}{l} x(t)=Ae^{p_1t}+Be^{p_2t}+\frac{1}{c} \\ \dot{x}(t)=p_1Ae^{p_1t}+p_2Be^{p_2t}\end{array}\right\} \Rightarrow \left\{\begin{array}{l} x(0)=0=A+B+\frac{1}{c} \\ \dot{x}(0)=0=p_1A+p_2B\end{array}\right\}$$

$$\{0=p_1A+p_2B\} \quad\Rightarrow\quad \left\{A=\frac{-p_2}{p_1}B\right\};\quad (\frac{p_2}{p_1}-1)B=\frac{1}{c};\quad A=\frac{-p_2}{c(p_2-p_1)};\quad B=\frac{p_1}{c(p_2-p_1)}$$

Response includes two exponentials.

1. The exponential with the slowest time scale (larger p_i) has the largest coefficient.
2. If $p_2>>p_1$, then the response can be approximated by the slower exponential only.

$$x=\frac{1}{c}\left[1+\left(\frac{1}{p_2-p_1}\right)(p_1e^{p_2t}-p_2e^{p_1t})\right]$$

EXAMPLE 1 $\ddot{x}+3\dot{x}+2x=1$ $p^2+3p+2=0$ $\Rightarrow \quad p_1=-1,\ p_2=-2$ Plug in the solution from above.	$x=\frac{1}{c}\left[1+\left(\frac{1}{p_2-p_1}\right)(p_1e^{p_2t}-p_2e^{p_1t})\right]$ $p_1=-1, \quad p_2=-2, \quad c=2$ $x=\frac{1}{2}\left[1+\left(\frac{1}{-1}\right)(-e^{-2t}+2e^{-t})\right]$
EXAMPLE 2 $\ddot{x}+8\dot{x}+7x=14$ $p^2+8p+7=0$ $\Rightarrow \quad p_1=-7,\ p_2=-1$ Plug in the solution from above.	$x=\frac{14}{c}\left[1+\left(\frac{1}{p_2-p_1}\right)(p_1e^{p_2t}-p_2e^{p_1t})\right]$ $p_1=-7, \quad p_2=-1, \quad c=7$ $x=\frac{14}{7}\left[1+\left(\frac{1}{-1+7}\right)(-7e^{-t}+e^{-7t})\right]$

Note that slower mode is dominating

REMARK

With non-zero initial conditions the solution will not have such a simple generic solution. However, such scenarios with more involved algebra, offer little useful insight so are not pursued here.