Modelling and control summaries

by Anthony Rossiter

2nd order responses 10: sketching

Assume a standard 2nd order model with constant coefficients, output x(t) and input u(t). What is the expected impact on overshoot/oscillation of a change in damping ratio?	$\frac{d^2x}{dt^2} + 2\varsigma w_n \frac{dx}{dt} + w_n^2 x = f$
This resource assumes that damping ζ is less than one.	

Objective is to form a sketch of a 2nd order response with pen and paper which captures the key characteristics.

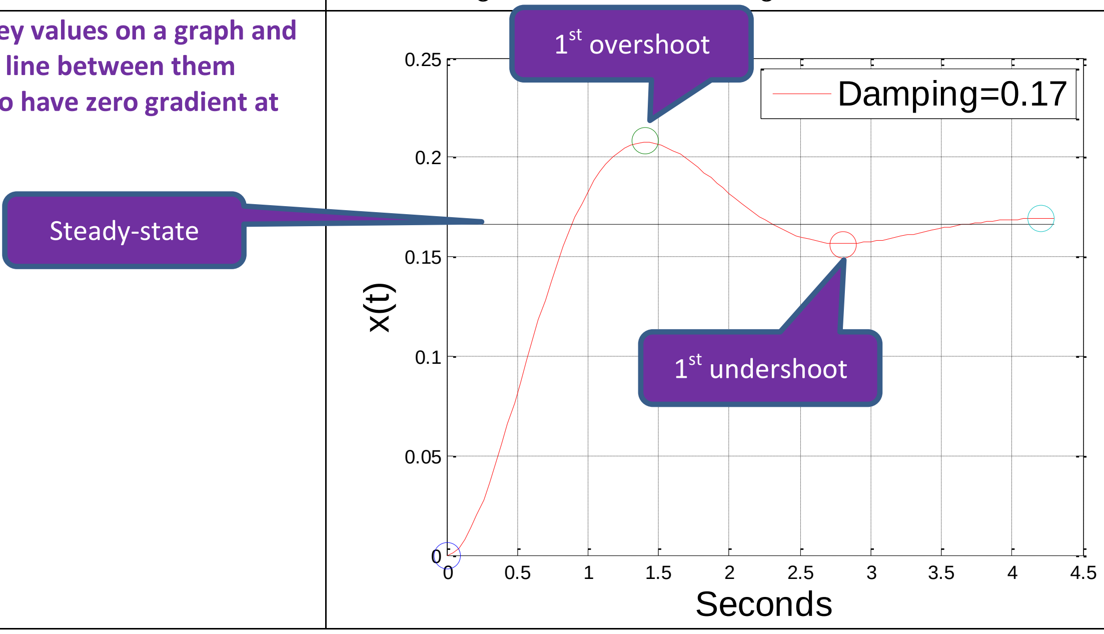
- It is conventional to do the sketch from zero initial conditions as other assumptions would required excessive reworking of the algebra so a quick sketch is not possible.
- The basic technique is to compute core values, put these on a graph and then sketch between them.

KEY CHARACTERISTICS FOR SKETCH

First overshoot time and subsequent overshoot/undershoot times	Magnitude of 1st overshoot (and subsequent overshoots and undershoots) in percentage.
$t = \frac{n\pi}{w_d} = \frac{n\pi}{w_n\sqrt{1-\varsigma^2}}, \quad n = 1, 2, \cdots$	$p = e^{\frac{-\varsigma\pi}{\sqrt{1-\varsigma^2}}}; \quad p^2, p^3,$
Steady-state and actual overshoots	Initial gradient
This is solved from $w_n^2 x = f \quad \Rightarrow \quad x_{ss} = \frac{f}{w_n^2}$ $x(t) = x_{ss}(1+p), \; x(2t) = x_{ss}(1-p^2)$ $x(3t) = x_{ss}(1+p^3), \cdots$	For zero initial conditions known to be zero.

EXAMPLE 1 - sketch the response for $\ddot{x} + 2\dot{x} + 6x = 1$

STEP 1: Put into normalised form and find the damping ratio and natural frequency	$\ddot{x} + 2\varsigma w_n \dot{x} + w_n^2 x = 1$ $\Rightarrow \quad w_n = \sqrt{6}; \quad \varsigma = \frac{1}{\sqrt{6}}$
STEP 2: Peak overshoot times.	$t = \frac{n\pi}{w_n\sqrt{1-\varsigma^2}} = 1.4, 2.8, \cdots$
STEP 3: Peak overshoot values (%)	$p = 0.25, \quad p^2 = 0.25^2 \approx 0.06, \cdots$

STEP 4: Steady-state and actual overshoot	$\lim_{t\to\infty} x(t) = x_{ss} = \frac{1}{6};$ $x(1.4) = \frac{1}{6}(1.25), \quad x(2.8) = \frac{1}{6}(0.94)$
STEP 5: Mark key values on a graph and draw a smooth line between them remembering to have zero gradient at t=0.	

EXAMPLE 2 for student work

Sketch the response for $\ddot{x} + 2\dot{x} + 17x = 25.5$ **and hence validate the sketch provided here**