Modelling and control summaries

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2nd order responses 4: under-damped systems

Assume a standard 2nd order model with constant coefficients, output y(t) and input u(t). What is the expected behaviour and how does this vary with a,b,c when the system has oscillatory modes?	$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f$
This resource focuses on the use of traditional methods to find a solution for x(t) and assumes zero initial conditions for convenience. Assume f(t) is a constant.	

An under - damped system is where the characteristic polynomial has complex roots	
$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f \quad \Rightarrow \quad ap^2+bp+c=0 \quad \Rightarrow \quad p=\alpha \pm j\beta$	
Complex roots p_1,p_2 implies that $b^2<4ac$. This means that, for constant f, the solution takes the following alternative and equivalent forms. The user decides which form is most convenient based on the context.	$x=\frac{f}{c}+Ae^{-\alpha t}\sin(\beta t+\phi)=\frac{f}{c}+Be^{-\alpha t}\cos(\beta t+\gamma)$ $x=\frac{f}{c}+e^{-\alpha t}[C\sin(\beta t)+D\cos(\beta t)]$
What remains is to determine the unknown coefficients of which there are two for each possible form.	

Example 1 with zero initial conditions (x(0)=0, dx/dt(0)=0)	
$\ddot{x}+2\dot{x}+2x=1$ $p^2+2p+2=0 \quad \Rightarrow \quad p=-1\pm j$	$x=\frac{1}{2}+Ae^{-t}\sin(t+\phi) \quad or \quad \frac{1}{2}+Be^{-t}\cos(t+\gamma)$
Substitute the initial conditions into the general solution.	$x(0)=0=\frac{1}{2}+Ae^{-0}\sin(0+\phi)$ $\dot{x}(0)=0=-Ae^{-0}\sin(0+\phi)+Ae^{-0}\cos(0+\phi)$
Solve the corresponding simultaneous equations.	$-0.5=A\sin(\phi); \quad A\sin(\phi)=A\cos(\phi) \quad \Rightarrow \quad \phi=\frac{\pi}{4}$ $A\sin(\frac{\pi}{4})=\frac{A}{\sqrt{2}}=-0.5 \quad \Rightarrow \quad A=\frac{-1}{\sqrt{2}}$
Hence:	$x=\frac{1}{2}-\frac{1}{\sqrt{2}}e^{-t}\sin(t+\frac{\pi}{4})$

Example 2 with zero initial conditions	
$\ddot{x}+6\dot{x}+10x=4$ $p^2+6p+10=0 \quad \Rightarrow \quad p=-3\pm j$	$x=\frac{4}{10}+Ae^{-3t}\sin(t+\phi)$
Substitute the initial conditions into the general solution.	$x(0)=0=\frac{4}{10}+Ae^{-0}\sin(0+\phi)$ $\dot{x}(0)=0=-3Ae^{-0}\sin(0+\phi)+Ae^{-0}\cos(0+\phi)$
Solve the corresponding simultaneous equations.	$-0.4=A\sin(\phi); \quad 3A\sin(\phi)=A\cos(\phi) \quad \Rightarrow \quad \tan\phi=\frac{1}{3}$ $\phi=0.32rad; \quad A\sin(\phi)=-0.4 \quad \Rightarrow \quad A=\frac{-0.4}{\sin(\varphi)}=-1.265$
Hence:	$x=0.4-1.265e^{-3t}\sin(t+0.32)$

SUMMARY

1. Second order ODES can be solved using a long approach as shown here.
2. With non-zero initial conditions the algebra will be a little more tricky.
3. As a general rule, the approach is somewhat cumbersome to do on pen and paper.

FURTHER PROBLEMS for readers to try

Solve the following and compare answers with solutions from MATLAB tools dsolve.m and ilaplace.m

$$\ddot{x}+6\dot{x}+10x=1;$$
$$\ddot{x}-4\dot{x}+20x=2;$$
$$5\ddot{x}+20\dot{x}+65x=10;$$
$$\ddot{x}+\dot{x}-2x=4$$
$$\ddot{x}+\dot{x}+4x=1$$