

Modelling and control summaries



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2nd order responses 5: under-damped & Laplace

Assume a standard 2nd order model with constant coefficients, output $y(t)$ and input $u(t)$. What is the expected behaviour and how does this vary with a, b, c when the system has oscillatory modes?

$$a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx = f$$

This resource focuses on the use of Laplace methods to find a solution for $x(t)$ and assumes zero initial conditions for convenience as these add no useful insight. Assume $f(t)$ is a constant.

Forming the Laplace transform (for convenience take $f(t)$ to be a unit step)

Take Laplace of every term in the equation and rearrange to find $x(s)$.

$$L[a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx] = L[f] \Rightarrow x(s)[as^2 + bs + c] = f(s) \Rightarrow x(s) = \frac{1}{as^2 + bs + c} \frac{1}{s}$$

The assumption of oscillatory modes means that the poles p_1, p_2 are complex ($b^2 < 4ac$).

Inverse Laplace methods: identify forms in the look-up table with oscillatory modes

The standard forms are summarised here.

A standard inverse Laplace technique is to express the given transform in the same forms that are available in the table.

$$L^{-1}\left[\frac{w}{[(s+p)^2 + w^2]}\right] = e^{-pt} \sin wt$$

$$L^{-1}\left[\frac{s+p}{[(s+p)^2 + w^2]}\right] = e^{-pt} \cos wt$$

For the given system this means that:

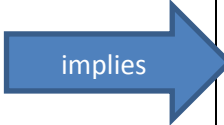
- for appropriate p, w .
- A, B, C , to be determined.

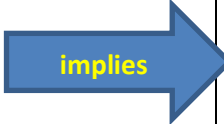
$$\frac{1}{[as^2 + bs + c]s} = \frac{A(s+p)}{[s+p]^2 + w^2} + \frac{Bw}{[s+p]^2 + w^2} + \frac{C}{s}$$

It should be clear that the poles (roots of $as^2 + bs + c = 0$) are given by $p_{1,2} = -p \pm jw$, $-p - jw$ so p and w are easy to find. For example (note how it is typical to make the pole polynomial monic first):

$$as^2 + bs + c = a\left(s^2 + \frac{b}{a}s + \frac{c}{a}\right) = a\left([s+p]^2 + w^2\right) \Rightarrow p = \frac{b}{2a}; \quad w = \sqrt{c - \frac{b^2}{4a^2}}$$

A, B, C are found using standard partial fraction methods.

Example 1 with zero initial conditions ($x(0)=0, dx/dt(0)=0$)	
$\ddot{x} + 2\dot{x} + 2x = 1;$ $s^2 + 2s + 2 = 0 \Rightarrow p = -1, w = 1$	 $x(s) = \frac{1}{(s+1)^2 + 1^2} \cdot \frac{1}{s}$
Express in partial fractions using appropriate forms from the table.	$\frac{1}{[(s+1)^2 + 1^2]s} = \frac{A(s+1)}{[s+1]^2 + 1^2} + \frac{B}{[s+1]^2 + 1^2} + \frac{C}{s}$
Use the cover-up rule to find C	C=0.5
Use expansion to find A,B. A=-0.5 and B=-0.5	$1 = (As + A + B)s + C(s^2 + 2s + 2)$ $A + C = 0, \quad A + B + 2C = 0$
From tables, x(t) is given as:	$x = C + Ae^{-t} \cos(t) + Be^{-t} \sin(t)$

Example 2 with zero initial conditions ($x(0)=0, dx/dt(0)=0$)	
$\ddot{x} + 4\dot{x} + 13x = 7;$ $s^2 + 4s + 13 = 0 \Rightarrow p = -2, w = 3$	 $x(s) = \frac{1}{(s+2)^2 + 3^2} \cdot \frac{7}{s}$
Express in partial fractions using appropriate forms from the table.	$\frac{7}{[(s+2)^2 + 3^2]s} = \frac{A(s+2)}{[s+2]^2 + 3^2} + \frac{3B}{[s+2]^2 + 3^2} + \frac{C}{s}$
Use the cover-up rule to find C	C=7/13
Use expansion to find A,B. A=-7/13 and B=-14/39	$7 = (As + 2A + 3B)s + C(s^2 + 4s + 13)$ $A + C = 0, \quad 2A + 3B + 4C = 0$
From tables, x(t) is given as:	$x = C + Ae^{-t} \cos(t) + Be^{-t} \sin(t)$ $x = \frac{7}{13} - \frac{7}{13} Ae^{-2t} \cos(3t) - \frac{14}{39} e^{-2t} \sin(3t)$

SUMMARY

1. Under-damped second order ODES can be solved using Laplace methods.
2. The main technique is to express the Transform as a sum of standard forms from the look-up table.
3. Standard partial fraction methods can be used to find the unknown coefficients.