Modelling and control summaries

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2nd order responses 5: under-damped & Laplace

Assume a standard 2nd order model with constant coefficients, output y(t) and input u(t). What is the expected behaviour and how does this vary with a,b,c when the system has oscillatory modes?	$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f$
This resource focuses on the use of Laplace methods to find a solution for x(t) and assumes zero initial conditions for convenience as these add no useful insight. Assume f(t) is a constant.	

Forming the Laplace transform (for convenience take f(t) to be a unit step)

Take Laplace of every term in the equation and rearrange to find x(s).

$$L[a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx]=L[f] \quad \Rightarrow \quad x(s)[as^2+bs+c]=f(s) \quad \Rightarrow \quad x(s)=\frac{1}{as^2+bs+c}\frac{1}{s}$$

The assumption of oscillatory modes means that the poles p_1,p_2 are complex ($b^2<4ac$).

Inverse Laplace methods: identify forms in the look-up table with oscillatory modes

The standard forms are summarised here.

A standard inverse Laplace technique is to express the given transform in the same forms that are available in the table.

$$L^{-1}[\frac{w}{[(s+p)^2+w^2]}]=e^{-pt}\sin wt$$

$$L^{-1}[\frac{s+p}{[(s+p)^2+w^2]}]=e^{-pt}\cos wt$$

For the given system this means that:

- for appropriate *p,w*.
- A,B,C, to be determined.

$$\frac{1}{[as^2+bs+c]s}=\frac{A(s+p)}{[s+p]^2+w^2}+\frac{Bw}{[s+p]^2+w^2}+\frac{C}{s}$$

It should be clear that the poles (roots of $as^2+bs+c=0$) are given by $p_{1,2}=-p+jw,-p-jw$ so p and w are easy to find. For example (note how it is typical to make the pole polynomial monic first):

$$as^2+bs+c=a(s^2+\frac{b}{a}s+\frac{c}{a})=a([s+p]^2+w^2) \quad \Rightarrow \quad p=\frac{b}{2a}; \quad w=\sqrt{c-\frac{b^2}{4a^2}}$$

A,B,C are found using standard partial fraction methods.

Example 1 with zero initial conditions (x(0)=0, dx/dt(0)=0)	
$\ddot{x}+2\dot{x}+2x=1$; $s^2+2s+2=0 \quad \Rightarrow \quad p=-1, w=1$ — implies	$x(s)=\dfrac{1}{(s+1)^2+1^2}\cdot\dfrac{1}{s}$
Express in partial fractions using appropriate forms from the table.	$\dfrac{1}{[(s+1)^2+1^2]s}=\dfrac{A(s+1)}{[s+1]^2+1^2}+\dfrac{B}{[s+1]^2+1^2}+\dfrac{C}{s}$
Use the cover-up rule to find C	C=0.5
Use expansion to find A,B. A=-0.5 and B=-0.5	$1=(As+A+B)s+C(s^2+2s+2)$ $A+C=0, \quad A+B+2C=0$
From tables, x(t) is given as:	$x=C+Ae^{-t}\cos(t)+Be^{-t}\sin(t)$

Example 2 with zero initial conditions (x(0)=0, dx/dt(0)=0)	
$\ddot{x}+4\dot{x}+13x=7$; $s^2+4s+13=0 \quad \Rightarrow \quad p=-2, w=3$ — implies	$x(s)=\dfrac{1}{(s+2)^2+3^2}\cdot\dfrac{7}{s}$
Express in partial fractions using appropriate forms from the table.	$\dfrac{7}{[(s+2)^2+3^2]s}=\dfrac{A(s+2)}{[s+2]^2+3^2}+\dfrac{3B}{[s+2]^2+3^2}+\dfrac{C}{s}$
Use the cover-up rule to find C	C=7/13
Use expansion to find A,B. A=-7/13 and B=-14/39	$7=(As+2A+3B)s+C(s^2+4s+13)$ $A+C=0, \quad 2A+3B+4C=0$
From tables, x(t) is given as:	$x=C+Ae^{-t}\cos(t)+Be^{-t}\sin(t)$ $x=\dfrac{7}{13}-\dfrac{7}{13}Ae^{-2t}\cos(3t)-\dfrac{14}{39}e^{-2t}\sin(3t)$

SUMMARY

1. Under-damped second order ODES can be solved using Laplace methods.
2. The main technique is to express the Transform as a sum of standard forms from the look-up table.
3. Standard partial fraction methods can be used to find the unknown coefficients.