Modelling and control summaries

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2nd order responses 6: normalised forms

Assume a standard 2nd order model with constant coefficients, output y(t) and input u(t). What is the expected behaviour and how does this vary with a,b,c when the system has oscillatory modes?	$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f$

This resource focuses on the use normalised forms to understand the behaviour. Normalised forms are most commonly used when the system has oscillatory behaviour, but can also give some insight to the over-damped case.

Definition of normalised form

First the ODE is expressed in monic form, that is the coefficient of the maximum power is one. Then toe coefficients of the 1st derivative and state are expressed in a particular format as show here.

$$\left\{a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f\right\} \equiv \left\{\frac{d^2x}{dt^2}+2\varsigma w_n\frac{dx}{dt}+w_n^2x=\frac{1}{a}f\right\}; \quad 2\varsigma w_n=\frac{b}{a}; \quad w_n^2=\frac{c}{a}$$

While these definitions may seem arbitrary and odd, it will be seen that there are clear links between the new parameters of ζ, w_n and the behaviours and thus this form gives good insight into the system.

- The coefficient ζ is referred to as the **damping factor** (or often just damping).
- The coefficient w_n is referred to as the **natural frequency** (the frequency at which the system would oscillate if the damping were zero).

Over-damped or underdamped behaviour

If the roots are complex, the behaviours is underdamped and if real, the behaviour is overdamped. It is straightforward to demonstrate that this distinction is captured by the magnitude of ζ.

$$s^2+2\varsigma w_n s+w_n^2=0 \quad \Rightarrow$$

When ζ>1 then the poles are real. → $p_{1,2}=-\varsigma w_n \pm w_n\sqrt{\varsigma^2-1}; \quad \varsigma>1$

When ζ<1 then the poles are complex. → $p_{1,2}=-\varsigma w_n \pm jw_n\sqrt{1-\varsigma^2}; \quad \varsigma<1$

REMARK: It is not normal to consider divergent/unstable behaviours with the normalised form. These would correspond to negative damping.

REMARK 2: The actual frequency of oscillation (if underdamped) is given by $w_d=w_n\sqrt{1-\varsigma^2}$

Examples of damping computations

CRITICALLY DAMPED	SLIGHTLY UNDERDAMPED
$\ddot{x}+2\dot{x}+x=1 \quad \Rightarrow \quad 2=2\varsigma w_n; \quad 1=w_n^2$ $\Rightarrow \quad \varsigma=1$	$\ddot{x}+6\dot{x}+10x=4 \quad \Rightarrow \quad 6=2\varsigma w_n; \quad 10=w_n^2$ $\Rightarrow \quad \varsigma=\frac{6}{2\sqrt{10}}=0.95$

OVER DAMPED	UNDER DAMPED
$\ddot{x} + 5\dot{x} + 2x = 1 \quad \Rightarrow \quad 5 = 2\varsigma w_n; \quad 2 = w_n^2$ $\Rightarrow \quad \varsigma = \frac{5}{2\sqrt{2}} = 1.77$	$\ddot{x} + \dot{x} + 2x = 1 \quad \Rightarrow \quad 1 = 2\varsigma w_n; \quad 2 = w_n^2$ $\Rightarrow \quad \varsigma = \frac{1}{2\sqrt{2}} = 0.35$

Significance of damping ratio

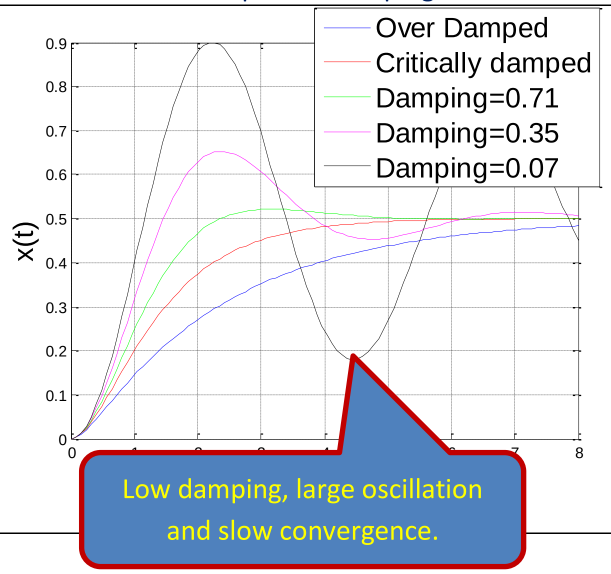
The damping ratio determines the shape of the response:

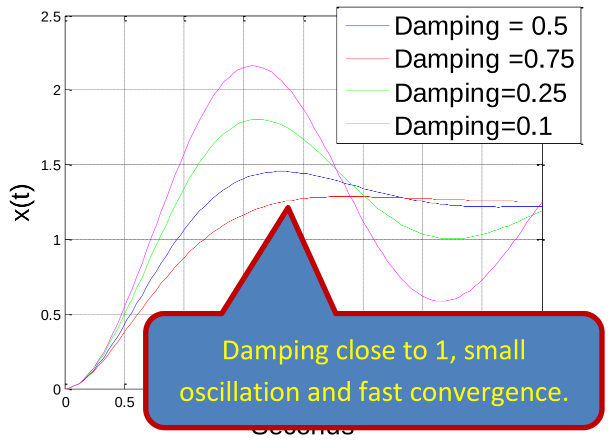
1. If this is close to zero, oscillation will dominate decay.
2. If this is close to one, decay will dominate oscillation.
3. If this is very large, the system will be over-damped and a slow mode will dominate.
4. In system design, engineers often look for a damping ratio bigger than about 0.7 as any less implies too much oscillation.

ζ >1 - an over damped system no oscillations
ζ =1 - a critically damped system no oscillations
ζ <1 - an under damped, oscillatory system

$$\frac{d^2x}{dt^2} + 2\varsigma w_n \frac{dx}{dt} + w_n^2 x = \frac{w_n^2}{2}$$

Illustrations of the impact of damping





SUMMARY

1. Under-damped second order ODES can be expressed in a normalised form.
2. Convergence rate and oscillation are closely linking to the damping ratio is this form.