Modelling and control summaries

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2nd order responses 8: convergence & damping

Assume a standard 2nd order model with constant coefficients, output x(t) and input u(t). What is the expected impact on covnergence rates of a change in damping ratio?	$\frac{d^2x}{dt^2} + 2\varsigma w_n \frac{dx}{dt} + w_n^2 x = f$

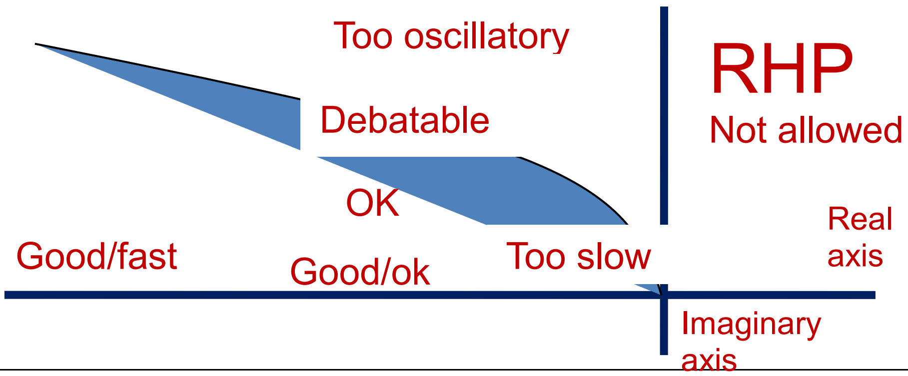
This resource assumes that the state converges so the limit $\lim_{t\to\infty} x(t)$ exists.

For convenience, assume that f, w_n are fixed so that consideration is given solely to the impact of damping. This is a reasonable assumption as it implies the steady-state is constant.

Assumption

Speed of convergence is linked directly to the pole positions (see resources on behaviours).

- A pole further into the LHP means faster convergence
- A pole closer to the imaginary axis means slower convergence.



Illustrations that pole positions change dramatically with damping

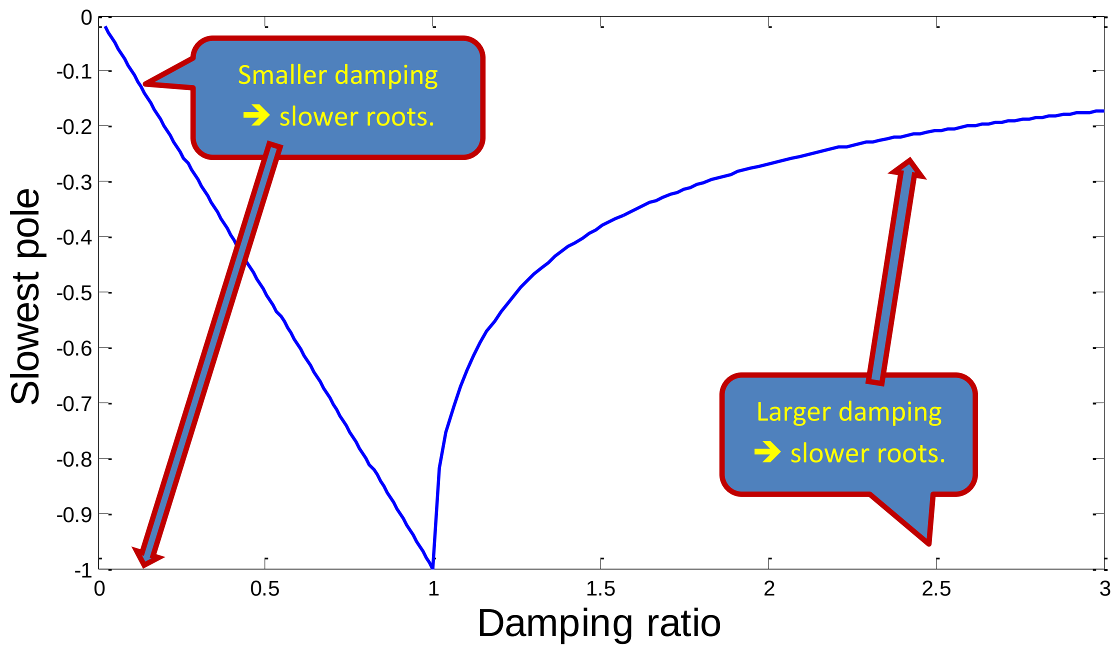
Damping ratio	Pole positions	
$\ddot{x} + 5\dot{x} + 2x = 1, \quad \varsigma = 1.77$	$-4.56, -0.43$	Fast and slow
$\ddot{x} + 3\dot{x} + 2x = 1, \quad \varsigma = 1.06$	$-2, -1$	
$\ddot{x} + 2\dot{x} + 2x = 1, \quad \varsigma = 0.71$	$-1 \pm j$	Both fast
$\ddot{x} + \dot{x} + 2x = 1, \quad \varsigma = 0.35$	$-0.5 \pm j1.3$	
$\ddot{x} + 0.2\dot{x} + 2x = 1, \quad \varsigma = 0.07$	$-0.1 \pm j1.41$	Both slow

The challenge is to discern whether there is a pattern to these changes.
If there is a pattern, it is likely we can exploit this for insight and design.

Root calculations

Under damped examples	Over damped examples
If ζ<1, the poles are complex then	If ζ>1, the poles are real then
$p^2 + 2\varsigma w_n p + w_n^2 = 0 \quad \Rightarrow \quad p = -\varsigma w_n \pm j w_n \sqrt{1-\varsigma^2}$	$p^2 + 2\varsigma w_n p + w_n^2 = 0 \quad \Rightarrow \quad p = -\varsigma w_n \pm w_n \sqrt{\varsigma^2 - 1}$
• The real part of the poles is -ζwn • As ζ gets smaller, the pole moves closer to the origin and thus the response slows down. **SMALLER DAMPING ➔ slower convergence**	• There are two poles, the one closest to the origin is at $w_n(-\varsigma + \sqrt{\varsigma^2 - 1})$ • As ζ gets bigger, this pole moves closer to the origin and thus the response slows down. **LARGER DAMPING ➔ slower convergence**

Example here uses w_n=1 and plots the real part of the slowest pole.



SUMMARY

1. If damping is less than one, reducing damping slows down convergence.
2. If damping is greater than one, increasing damping slows down convergence.
3. Faster convergence corresponds to damping ratios around 1.

QUESTION

Which of the following will converge to the steady-state more quickly and why?

$$\ddot{x} + 5\dot{x} + 16x = 1; \quad \ddot{x} + 8\dot{x} + 16x = 1$$

$$\ddot{x} + 20\dot{x} + 16x = 1; \quad \ddot{x} + 12\dot{x} + 16x = 1$$