Modelling and control summaries

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2nd order responses 9: overshoot & damping

Assume a standard 2nd order model with constant coefficients, output x(t) and input u(t). What is the expected impact on overshoot/oscillation of a change in damping ratio?	$\frac{d^2x}{dt^2}+2\varsigma w_n\frac{dx}{dt}+w_n^2x=f$

This resource assumes that damping is less than one and uses $f=w_n^2$ for convenience.

Assumption

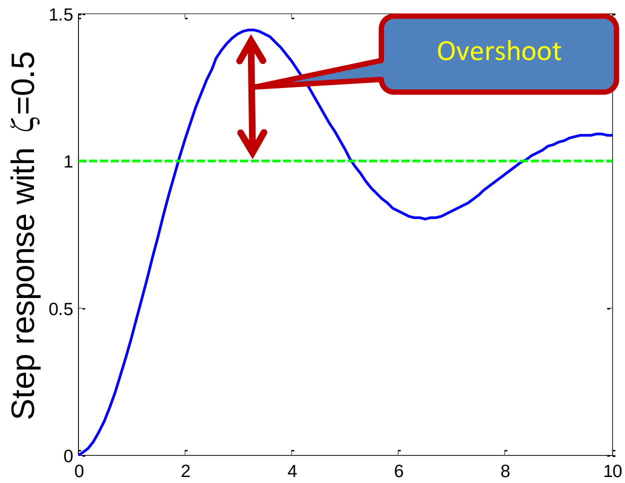
We know that with an underdamped system there is oscillation. We want to know what the impact of this is on behaviour – a step response from zero initial conditions is used as a good indicator.

For a step response which reflects common scenarios in industry, we are particularly interested in the first overshoot as large overshoots can be very damaging in practice.

If ζ<1, the poles are given by:

$$p^2+2\varsigma w_n p+w_n^2=0 \quad\Rightarrow\quad p=-\varsigma w_n \pm j w_n\sqrt{1-\varsigma^2}$$

It is notable that the frequency gets faster as damping reduces, but reaches a limit of w_n for ζ=0.



General solution for step response of an underdamped 2nd order ODE (zero initial conditions)

Only an abbreviated derivation is given here.

$$\frac{d^2x}{dt^2}+2\varsigma w_n\frac{dx}{dt}+w_n^2x=w_n^2 \quad\Rightarrow\quad x=Ae^{-\varsigma w_n t}\sin(w_d t+\phi)+C \qquad \left\{w_d=w_n\sqrt{1-\varsigma^2}\right\}$$

Substitute in the initial conditions and steady-state $\lim_{t\to\infty}x(t)$ =1.

$$x(t)=1-Ae^{-\varsigma wt}\sin(w_d t+\phi); \quad \frac{dx}{dt}=Ae^{-\varsigma w_n t}[-w_d\cos(w_d t+\phi)+\varsigma w_n\sin(w_d t+\phi)]$$

$$x(0)=0 \quad\Rightarrow\quad 0=1-A\sin\phi; \quad \dot{x}(0)=0 \quad\Rightarrow\quad -w_d\cos(\phi)+\varsigma w\sin(\phi)=0$$

HENCE: $$\frac{\sin\phi}{\cos\phi}=\tan\phi=\frac{w_d}{\varsigma w_n}=\frac{\sqrt{1-\varsigma^2}}{\varsigma}; \quad A=\frac{1}{\sin\phi}=\frac{1}{\sqrt{1-\varsigma^2}}$$

AND THUS: $$x(t)=1-\frac{e^{-\varsigma w_n t}}{\sqrt{1-\varsigma^2}}\sin\left(w_d t+\tan^{-1}\frac{\sqrt{1-\varsigma^2}}{\varsigma}\right)$$

Interpreting the general step response solution

Peak overshoot time

The peak overshoot occurs when the derivative is zero.

- We already know that the derivative is zero at t=0.
- Hence the next zero point must occur when $w_d t = \pi$

$$t = \frac{\pi}{w_d} = \frac{\pi}{w_n\sqrt{1-\varsigma^2}}$$

Peak overshoot value

The magnitude of the overshoot is determined by substituting the peak time into the general solution for x(t) above.

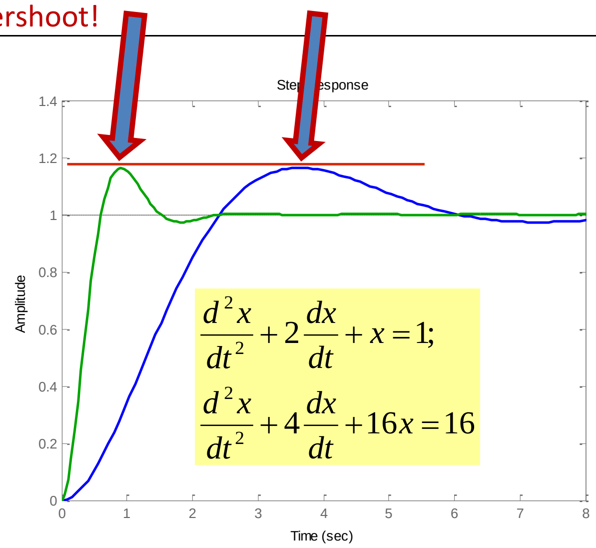
$$w_n t = \frac{\pi}{\sqrt{1-\varsigma^2}}; \quad x(t) = 1 - \frac{e^{-\varsigma\pi/\sqrt{1-\varsigma^2}}\sin(\pi+\phi)}{\sqrt{1-\varsigma^2}};$$

$$overshoot = x(t) - 1 = \frac{e^{-\varsigma\pi/\sqrt{1-\varsigma^2}}\sin(\phi)}{\sqrt{1-\varsigma^2}}$$

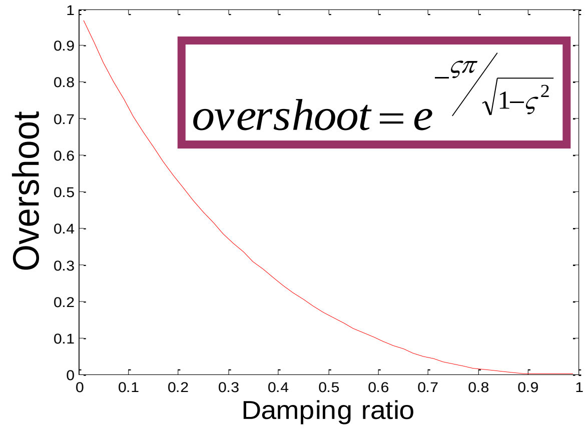
$$but \quad \frac{\sin(\phi)}{\sqrt{1-\varsigma^2}} = 1 \quad \Rightarrow \quad overshoot = e^{-\varsigma\pi/\sqrt{1-\varsigma^2}}$$

REMARK: The overshoot formulae is independent of w_n, it depends solely on the damping ratio!

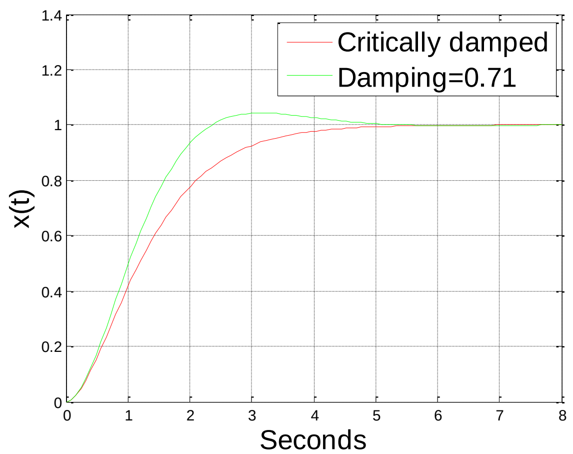
Same damping (here ζ=0.5) implies same overshoot!



Overshoot has simple dependence on damping



Overshoot becomes negligible for ζ>0.7



Successive overshoots follow the same ratio of convergence