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State-space analysis 1 introduction

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Introduction

- The first two sections looked at the definition of state space models and the computation of underlying behaviours.

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- The next job is to analyse the behaviours more carefully.
- This series introduces concepts of stability, controllability and observability.

Stability

Stability is a word used a lot in control and often it is used rather carelessly.

Here, an attempt is made to give some more rigorous definitions although in everyday parlance these are often described with a single expression of stability.

Convergence/stable equilibrium	Bounded input bounded output/state BIBO/BIBS

Equilibrium

One core test is whether there exists an equilibrium point such that a state remains at this point once there, and also (for stability), converges to that point from anywhere nearby.

Definitions for equilibrium are obvious, in that the state must not move, hence:

$$\dot{x} = 0 = Ax + Bu; \qquad x_{k+1} = Ax_k + Bu_k = x_k$$

Different constant values for u give different equilibria points although for convenience we will often take u=0.

Stable equilibrium

This is defined using norms and establishes that once the state is within a certain distance of the equilibrium point (use origin for convenience), thereafter it can never go further away again than a specified relative distance.

- This need not imply convergence, but it does ensure boundedness.
- ‘r’ is a scalar factor to be determined which for the linear case is independent of the magnitude of ‘x’.

$$\|x(t)\| \le \delta \quad \Rightarrow \quad \|x(t+t_1)\| \le r\delta, \quad \forall t_1 > 0$$

This case allows for permanent oscillation, so no convergence.

Asymptotically stable equilibrium

This is similar to stable equilibrium but with the difference that the one insists the state converges to the equilibrium point. E.g.

$$\lim_{t\to\infty} \|x(t)\| = 0$$

For non-linear systems there are some nuances not discussed here.
Pure oscillatory modes/sinusoids are not possible here.

Bounded input bounded state

This means that, assuming the input is bounded beneath some value, one can guarantee that the state is also bounded beneath another norm.

$$\left\{\|u(t)\| \le \delta, \forall t > 0\right\} \quad \Rightarrow \quad \left\{\|x(t)\| \le \varepsilon, \forall t > 0\right\}$$

In general the norms may depend upon the initial condition, but for linear systems, one could make these relationships more precise if desired.

The key point is that the state is bounded, it cannot diverge to infinity.

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Bounded input bounded output

This is slightly more relaxed than bounded state because:

$$y = Cx$$

It is possible for y to be bounded and x to be unbounded, as long as the unbounded part of x lies in the kernel of matrix C.

This definition illustrates a danger, that is observing or guaranteeing stability of the output need not imply stability of the system states.

Linear state space systems

Next we consider the implications of these definitions on linear state space systems.

For convenience we can separate the analysis into:

- Forced mode: input is non zero. This uses the BIBS/BIBO analysis.
- Free mode: input is zero. This uses the asymptotic stability analysis.

Reminder of system behaviour

Given a model and constant u.

$$\dot{x} = Ax + Bu; \quad y = Cx$$

$$eigenvalues \equiv |\lambda I - A| = 0$$

$$x(t) = k + w_1 e^{\lambda_1 t} + w_2 e^{\lambda_2 t} + \cdots + w_n e^{\lambda_n t}$$

Next, we analyse x(t) using the stability criteria.

Free mode

When the input is zero, then:

$$x(t) = w_1 e^{\lambda_1 t} + w_2 e^{\lambda_2 t} + \cdots + w_n e^{\lambda_n t}$$

Asymptotic stability requires that all the exponentials have negative real exponents:

$$\left\{ \lim_{t\to\infty} \|x(t)\| = 0 \right\} \quad \Rightarrow \quad \left\{ \mathrm{Re}(\lambda_i) < 0, \forall i \right\}$$

Pure oscillation is allowed, and hence, just stability is achieved if, for some δ:

$$\left\{ \underbrace{\max}_{t} \|x(t)\| < \delta \|x(0)\| \right\} \quad \Rightarrow \quad \left\{ \mathrm{Re}(\lambda_i) \le 0, \forall i \right\}$$

Forced mode

When the input is non-zero then **USUALLY**:

$$x(t) \approx ku + w_1 e^{\lambda_1 t} + w_2 e^{\lambda_2 t} + \cdots + w_n e^{\lambda_n t}$$

Where ‘ku’ represents the input mode being replicated in the output dynamics (precise details not required):

Nevertheless, BIBS requires asymptotic stability.

Eigenvalues must be strictly in the LHP because if a sinusoidal input mode matches an eigenvalue on the imaginary axis, state divergence could result (resonance/repeated pole).

$$\left\{ u(s) = \frac{1}{s^2 + w^2}; \quad G(s) = \frac{1}{s^2 + w^2} \right\} \Rightarrow Gu \rightarrow t \sin wt$$

Illustration of modes unaffected by u

The forced response is given by:

$$x = \int_0^t e^{A(t-\tau)} B u \, d\tau = \sum_i \int_0^t w_i e^{\lambda_i (t-\tau)} \underbrace{v_i^T B}_{\beta_i^T} u(\tau) \, d\tau$$

$$x = \sum_i \int_0^t w_i e^{\lambda_i (t-\tau)} \beta_i^T u(\tau) \, d\tau$$

Clearly, if $\beta_i = 0$, then the associated mode is not excited and hence BIBS can be achieved with the corresponding eigenvalue on the imaginary axis.

BIBO stability

The forced response is given by:

$$y=\int_0^t Ce^{A(t-\tau)}Bud\tau=\sum_i\int_0^t \underbrace{Cw_i}_{\gamma_i}e^{\lambda_i(t-\tau)}\underbrace{v_i^T B}_{\beta_i^T}u(\tau)d\tau$$

$$y=\sum_i\int_o^t \gamma_i e^{\lambda_i(t-\tau)}\beta_i^T u(\tau)d\tau$$

Clearly, if either $\gamma_i = 0$ or $\beta_i = 0$, then the associated mode is not excited/observed and hence BIBO can be achieved with the corresponding eigenvalue on the imaginary axis rather than strictly in the LHP.

EXTENSIONS TO THE DISCRETE CASE

Reminder of system behaviour

Given a model and constant u.

$$\left.\begin{array}{r} x_{k+1} = Ax_k + Bu_k \\ u_k = u_{k+1} = u_{k+2} = \cdots \end{array}\right\} \Rightarrow \quad x_{k+n} = \Phi(n)x_k + H(n)u_k$$

$$\Phi(n) = A^n; \qquad H(n) = B + AB + \cdots + A^{n-1}B$$

For time varying u.

$$x(k+1) = \sum_{n=0,1,\cdots} A^n Bu(k-n) + w_1\lambda_1^n\alpha_1 + w_2\lambda_2^n\alpha_2 + \cdots + w_n\lambda_n^n\alpha_n$$

Free mode

When the input is zero, then:

$$x(n) = w_1\lambda_1^n\alpha_1 + w_2\lambda_2^n\alpha_2 + \cdots + w_n\lambda_n^n\alpha_n$$

Asymptotic stability requires that all the eigenvalues are strictly inside the unit circle.

$$\left\{\lim_{n\to\infty}\|x(n)\| = 0\right\} \Rightarrow \left\{|\lambda_i| < 1, \forall i\right\}$$

Pure oscillation is allowed, and hence, just stability is achieved if, for some eigenvalues lie on the unit circle:

$$\left\{\underbrace{\max}_{n}\|x(n)\| < \delta\|x(0)\|\right\} \Rightarrow \left\{|\lambda_i| \le 1, \forall i\right\}$$

Forced mode

When the input is non-zero then **USUALLY**:

$$x(k+1) = \sum_{n=0,1,\cdots} A^n Bu(k-n) + w_1\lambda_1^n\alpha_1 + w_2\lambda_2^n\alpha_2 + \cdots + w_n\lambda_n^n\alpha_n$$

Where the input mode is often replicated in the output dynamics (precise details not required):

Nevertheless, BIBS requires asymptotic stability.

Eigenvalues must be strictly inside the unit circle because if a sinusoidal input mode matches an eigenvalue on the unit circle, state divergence could result (**as noted with the continuous time case**).

Illustration of modes unaffected by u

The forced response is given by:

$$x(k+1) = \sum_{n=0,1,\cdots} A^n Bu(k-n) = \sum_n \sum_i w_i \lambda_i^n \underbrace{v_i^T B}_{\beta_i^T} u(k-n)$$

$$y(k+1) = \sum_{n=0,1,\cdots} CA^n Bu(k-n) = \sum_n \sum_i \underbrace{Cw_i}_{\gamma_i} \lambda_i^n \underbrace{v_i^T B}_{\beta_i^T} u(k-n)$$

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Clearly, if either γ_i =0 or β_i =0, then the associated mode is not excited/observed and hence BIBO can be achieved with the corresponding eigenvalue on the unit circle rather than strictly inside.

Summary

Introduced concepts of stability.

1. Noted that depending on the requirement, eigenvalues must either be strictly in the LHP or can lie on the imaginary axis for continuous time (equivalent definitions for discrete time).
2. Of particular note is that BIBS➔BIBO but not vice versa as not all state directions are observed in the output.
3. Similarly, asymptotic stability implies stability, but not vice versa.

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