State-space analysis 2 controllability

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Introduction

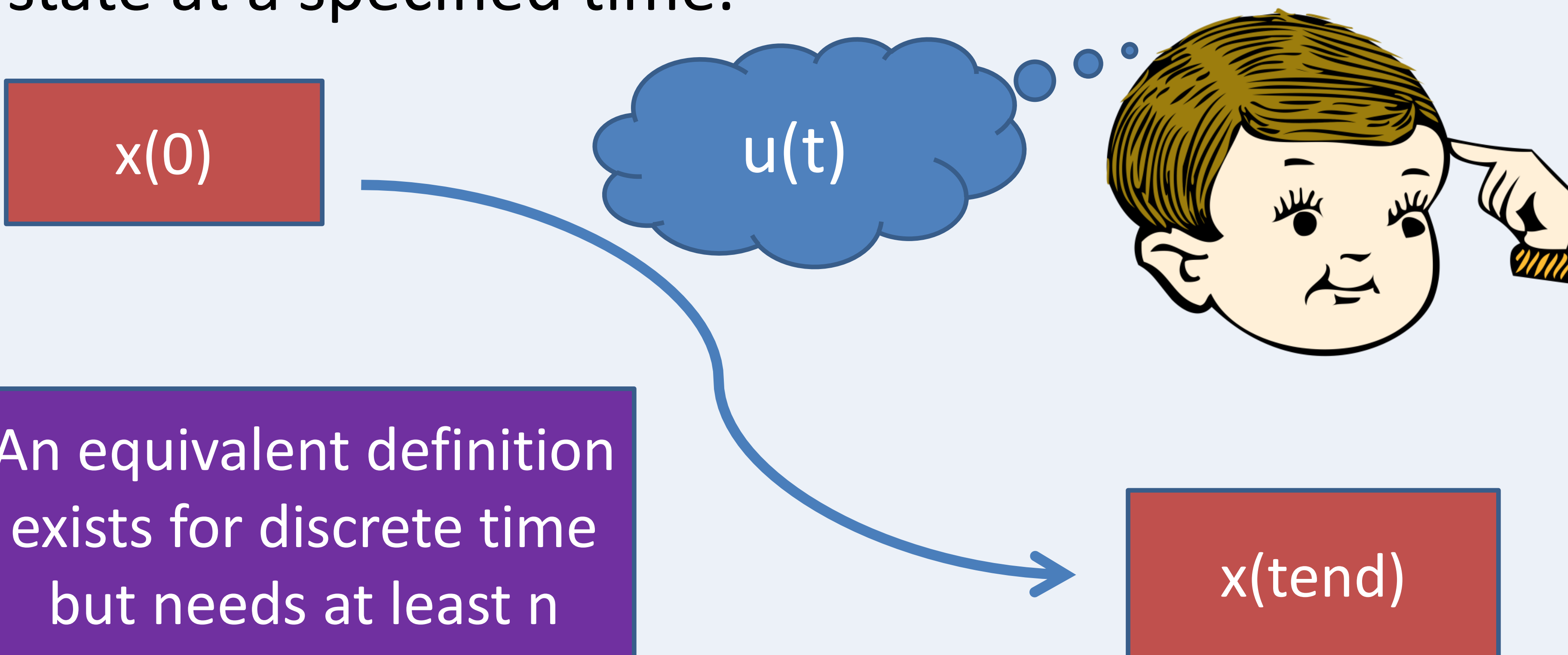
- The first video looked at definitions of stability, which is the inherent boundedness/convergence properties of a system response, with a bounded input.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- However, in general we are interested not just in convergence, but also the asymptotic value being the one we desire.
- Controllability links to the ability to achieve the desired asymptotic state.

Controllability (reachability)

A system is controllable if it is possible to determine an appropriate input signal, for any initial state, which will achieve a specified final state at a specified time.



An equivalent definition exists for discrete time but needs at least n samples.

Remark on Controllability

A system given in control canonical form is always fully controllable.

$$\frac{d}{dt}[z]=\underbrace{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u;$$

$$y=\underbrace{\begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}}_{C} z$$

Concepts of range

Let a vector x of dimension n be made up of a linear combination of 'n' other vectors w_i.

$$x = \alpha_1 w_1 + \alpha_2 w_2 + \cdots \alpha_n w_n$$

Does there exist choices of α_i such that x can take any value in n-dimensional space?

The answer is yes, if and only if the matrix W is full rank.

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = W^{-1} x$$

Controllability

A system is controllable if it is possible to determine an appropriate input signal which will achieve a specified final state x(t).

If x(t) can be represented as:

$$x(t) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots \alpha_n w_n$$

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$$

Then controllability reduces to checking the rank of matrix W and the ensuring one can choose the parameters α_i freely.

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Using eigenvalue/vector decomposition

This decomposition is useful for identifying whether there are regions in the state space which cannot be reached, from an arbitrary start point.

Without out loss of generality, we will assume the initial condition is zero!

$$x = \int_0^t e^{A(t-\tau)} Bu \, d\tau = \sum_i \int_0^t w_i e^{\lambda_i (t-\tau)} \underbrace{v_i^T B}_{\beta_i^T} u(\tau) d\tau$$

$$W = [w_1 \quad w_2 \quad \cdots \quad w_n]$$

W is full rank.

Using eigenvalue/vector decomposition

It is clear that x(t) has the structure indicated a few slides ago, that is:

$$x(t) = \sum_i w_i \int_0^t e^{\lambda_i (t-\tau)} \underbrace{v_i^T B}_{\beta_i^T} u(\tau) d\tau$$

$$x(t) = \sum_i w_i \alpha_i \qquad \alpha_i = \int e^{\lambda_i (t-\tau)} v_i^T B u(\tau) d\tau$$

It is clear that to choose x(t) arbitrarily 2 conditions are required:

- $\alpha_i \neq 0$, and indeed can be specified as required.
- W is full rank (**already known**).

Using eigenvalue/vector decomposition

We require that α_i can be chosen freely.

$$\alpha_i = \underbrace{v_i^T B}_{\beta_i} \int e^{\lambda_i (t-\tau)} u(\tau) d\tau$$

$$x = \sum_i w_i \alpha_i$$

Assuming that input u(t) is a free variable so that one can manage the output of the integration, then α_i can be chosen freely iff β_i is non-zero.

It is clear that $\beta_i=0 \rightarrow \alpha_i=0$, so a final state with a component in the corresponding eigenvector direction cannot be reached!

Using eigenvalue/vector decomposition

A system is fully controllable if and only if the VB matrix has no zero rows.

$$A = W\Lambda V; \quad \begin{bmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_n^T \end{bmatrix} = VB$$

It is clear that if $\beta_i=0$, then the forced mode has no contribution along the corresponding eigenvector direction!

If $\beta_i \neq 0$, then there always exists a choice of u(t) to define the required contribution along the corresponding eigenvector direction!

NUMERICAL EXAMPLES

Example 1

Is the following system controllable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

First do the eigenvalue/vector decomposition.

$$|\lambda I - A| = 0 \quad \Rightarrow \quad \lambda = 2, -1$$

$$(A - 2I)w_1 \quad \Rightarrow \quad \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A + I)w_2 \quad \Rightarrow \quad \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Example 1

The eigenvector matrix is therefore given as?

$$W = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}; \quad V = W^{-1} = \frac{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}}{3}$$

Therefore, one can find VB as:

$$VB = \frac{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}}{3} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.33 \\ 2.67 \end{bmatrix}$$

Clearly all rows non-zero so controllable.

Example 2

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

First do the eigenvalue/vector decomposition.

$$|\lambda I - A| = 0 \quad \Rightarrow \quad \lambda = 4.24, -0.24, 0$$

$$(A - \lambda_1 I)w_1 \quad \Rightarrow \quad w_1 = \begin{bmatrix} -0.6 \\ -0.78 \\ -0.18 \end{bmatrix}$$

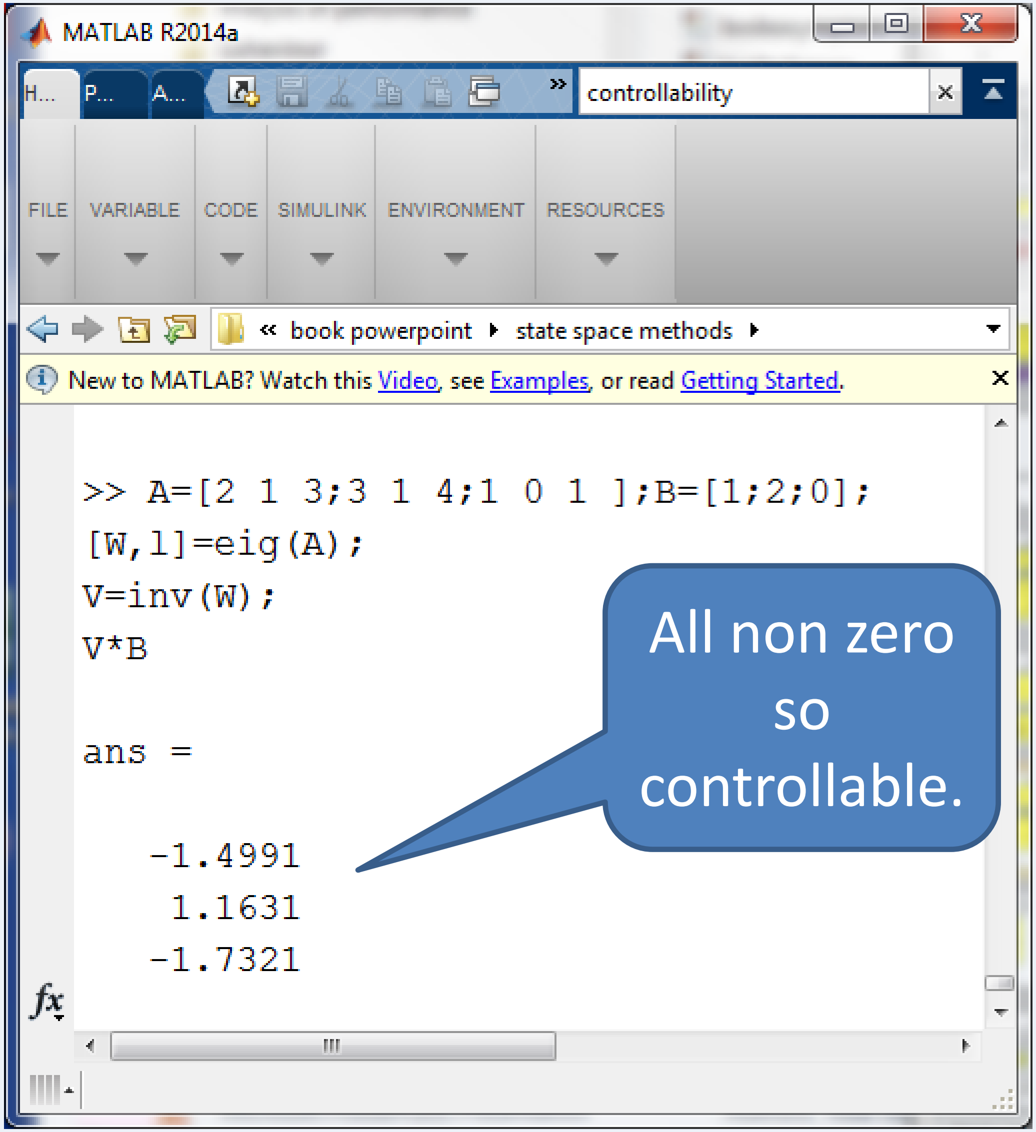
Not paper and pen computations so we revert to MATLAB to demonstrate more.

Example 2 continued

MATLAB is used to solve:

$$A = W \Lambda V;$$

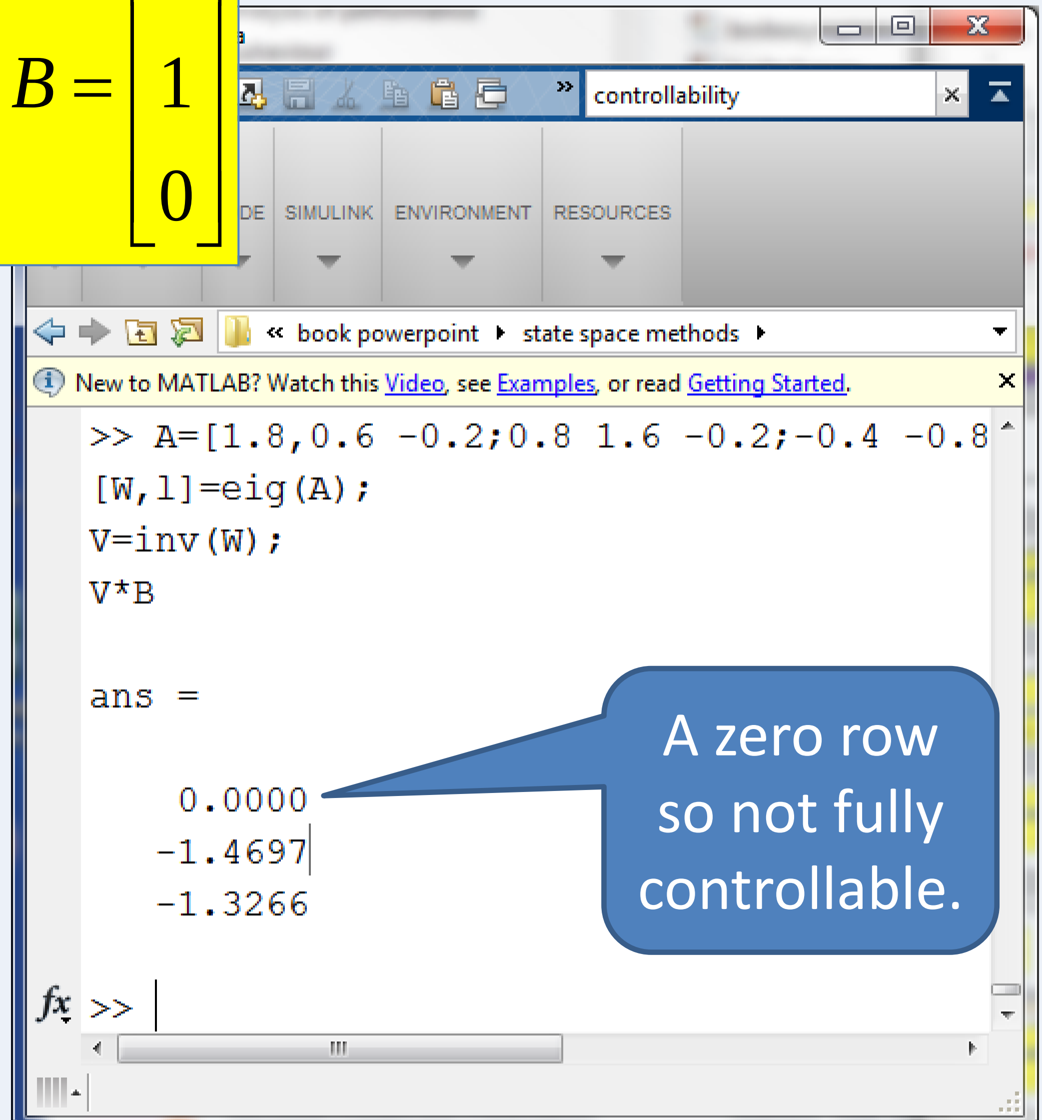
$$\begin{bmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_n^T \end{bmatrix} = VB$$



Example 3

$$A=\begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.4 & -0.8 & 2.6 \end{bmatrix}; \quad B=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Using MATLAB to compute the decomposition and VB matrix gives:



Summary

Used eigenvalue/vector decompositions to introduce concepts of controllability/reachability which means the ability to reach any desired value of x(t) by judicious choice of u(t).

$$\dot{x} = Ax + Bu$$

$$A = W\Lambda V$$

- Shown that full controllability requires the matrix VB to have no zero rows.
- Not discussed non-simple Jordan forms/systems with repeated eigenvalues.

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