



# State-space analysis 2 controllability

J A Rossiter

# Introduction

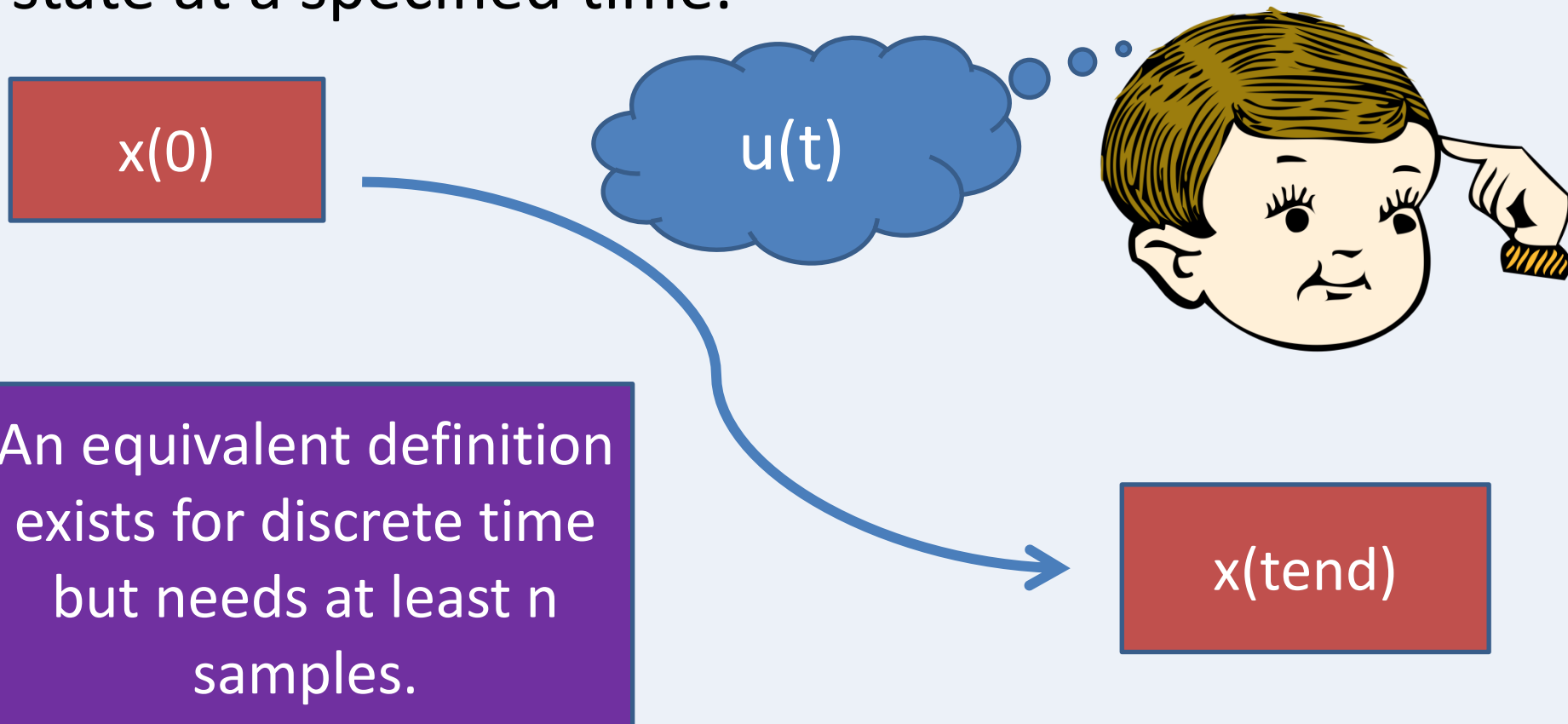
- The first video looked at definitions of stability, which is the inherent boundedness/convergence properties of a system response, with a bounded input.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- However, in general we are interested not just in convergence, but also the asymptotic value being the one we desire.
- Controllability links to the ability to achieve the desired asymptotic state.

# Controllability (reachability)

A system is controllable if it is possible to determine an appropriate input signal, for any initial state, which will achieve a specified final state at a specified time.



# Remark on Controllability

A system given in control canonical form is always fully controllable.

$$\frac{d}{dt} [z] = \underbrace{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_A z + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_B u;$$

$$y = \underbrace{\begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}}_C z$$

# Concepts of range

Let a vector  $x$  of dimension  $n$  be made up of a linear combination of ' $n$ ' other vectors  $w_i$ .

$$x = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$$

Does there exist choices of  $\alpha_i$  such that  $x$  can take any value in  $n$ -dimensional space?

The answer is yes, if and only if the matrix  $W$  is full rank.

$$W = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = W^{-1}x$$

# Controllability

A system is controllable if it is possible to determine an appropriate input signal which will achieve a specified final state  $x(t)$ .

If  $x(t)$  can be represented as:

$$x(t) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$

$$W = [w_1 \quad w_2 \quad \cdots \quad w_n]$$

Then controllability reduces to checking the rank of matrix  $W$  and the ensuring one can choose the parameters  $\alpha_i$  freely.

# Using eigenvalue/vector decomposition

This decomposition is useful for identifying whether there are regions in the state space which cannot be reached, from an arbitrary start point.

Without out loss of generality, we will assume the initial condition is zero!

$$x = \int_0^t e^{A(t-\tau)} B u d\tau = \sum_i \int_0^t w_i e^{\lambda_i(t-\tau)} \underbrace{v_i^T}_{\beta_i^T} B u(\tau) d\tau$$

$$W = [w_1 \quad w_2 \quad \cdots \quad w_n]$$

W is full rank.

# Using eigenvalue/vector decomposition

It is clear that  $x(t)$  has the structure indicated a few slides ago, that is:

$$x(t) = \sum_i w_i \int_0^t e^{\lambda_i(t-\tau)} \underbrace{v_i^T}_{\beta_i^T} B u(\tau) d\tau$$

$$x(t) = \sum_i w_i \alpha_i \quad \alpha_i = \int_0^t e^{\lambda_i(t-\tau)} v_i^T B u(\tau) d\tau$$

It is clear that to choose  $x(t)$  arbitrarily 2 conditions are required:

- $\alpha_i \neq 0$ , and indeed can be specified as required.
- $W$  is full rank (**already known**).



# Using eigenvalue/vector decomposition

We require that  $\alpha_i$  can be chosen freely.

$$\alpha_i = \underbrace{v_i^T B}_{\beta_i} \int e^{\lambda_i(t-\tau)} u(\tau) d\tau$$

$$x = \sum_i w_i \alpha_i$$

Assuming that input  $u(t)$  is a free variable so that one can manage the output of the integration, then  $\alpha_i$  can be chosen freely iff  $\beta_i$  is non-zero.

It is clear that  $\beta_i=0 \rightarrow \alpha_i=0$ , so a final state with a component in the corresponding eigenvector direction cannot be reached!

# Using eigenvalue/vector decomposition

A system is fully controllable if and only if the VB matrix has no zero rows.

$$A = W\Lambda V; \quad \begin{bmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_n^T \end{bmatrix} = VB$$

It is clear that if  $\beta_i=0$ , then the forced mode has no contribution along the corresponding eigenvector direction!

If  $\beta_i \neq 0$ , then there always exists a choice of  $u(t)$  to define the required contribution along the corresponding eigenvector direction!

# NUMERICAL EXAMPLES

# Example 1

Is the following system controllable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

First do the eigenvalue/vector decomposition.

$$|\lambda I - A| = 0 \Rightarrow \lambda = 2, -1$$

$$(A - 2I)w_1 \Rightarrow \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A + I)w_2 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

# Example 1

The eigenvector matrix is therefore given as?

$$W = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}; \quad V = W^{-1} = \frac{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}}{3}$$

Therefore, one can find VB as:

$$VB = \frac{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}}{3} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.33 \\ 2.67 \end{bmatrix}$$

Clearly all rows non-zero so controllable.

# Example 2

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

First do the eigenvalue/vector decomposition.

$$|\lambda I - A| = 0 \Rightarrow \lambda = 4.24, -0.24, 0$$

$$(A - \lambda_1 I)w_1 \Rightarrow w_1 = \begin{bmatrix} -0.6 \\ -0.78 \\ -0.18 \end{bmatrix}$$

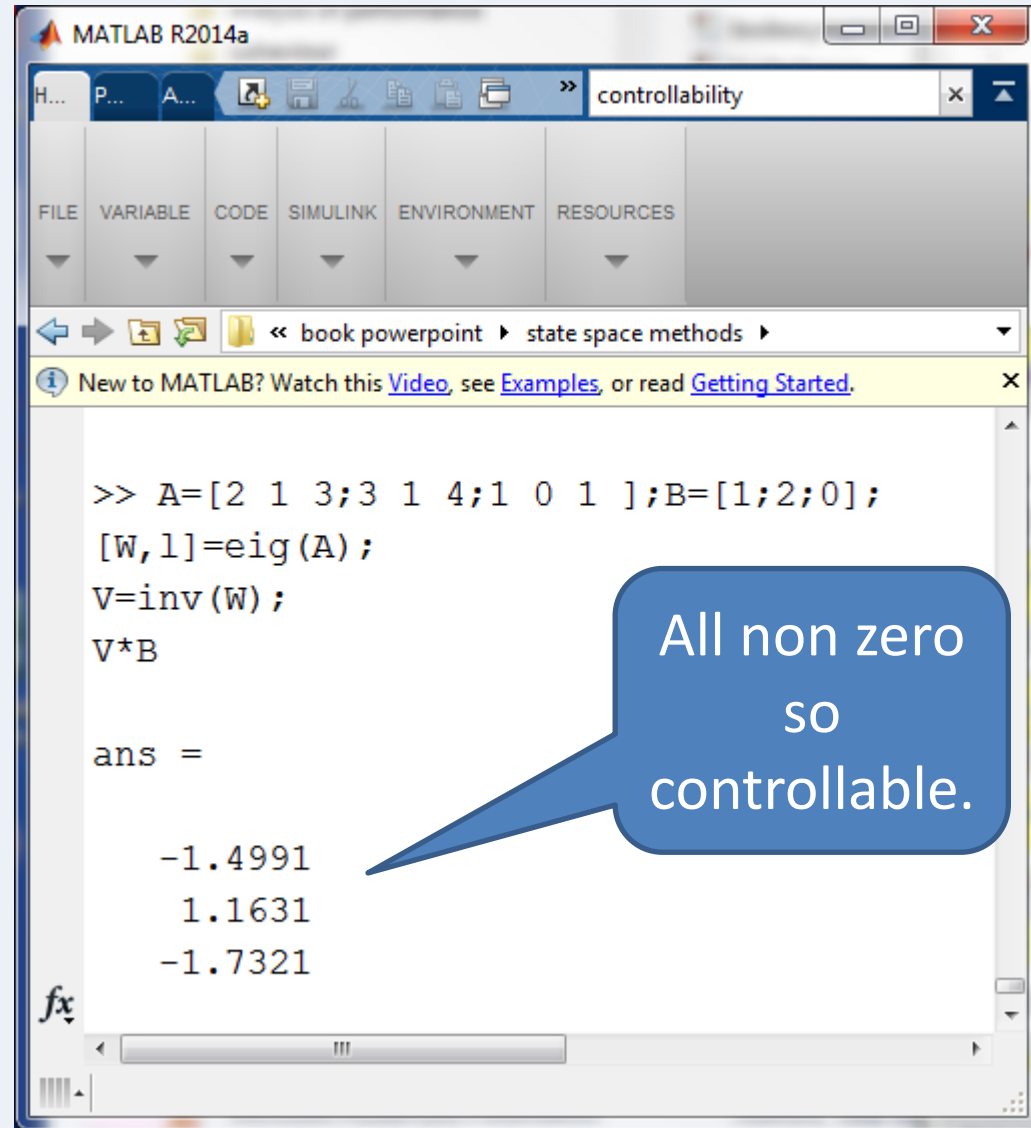
Not paper and pen computations so we revert to MATLAB to demonstrate more.

# Example 2 continued

MATLAB is used to solve:

$$A = W\Lambda V;$$

$$\begin{bmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_n^T \end{bmatrix} = VB$$



```

MATLAB R2014a
controllability

FILE VARIABLE CODE SIMULINK ENVIRONMENT RESOURCES

<< book powerpoint >> state space methods >

New to MATLAB? Watch this Video, see Examples, or read Getting Started.

>> A=[2 1 3;3 1 4;1 0 1 ];B=[1;2;0];
[W,l]=eig(A);
V=inv(W);
V*B

ans =

-1.4991
 1.1631
-1.7321
    
```

All non zero  
so  
controllable.

# Example 3

$$A = \begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.4 & -0.8 & 2.6 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Using MATLAB to compute the decomposition and VB matrix gives:

```

>> A=[1.8,0.6 -0.2;0.8 1.6 -0.2;-0.4 -0.8 2.6]
>> [W,l]=eig(A);
>> V=inv(W);
>> V*B

ans =

    0.0000
   -1.4697
   -1.3266
    
```

A zero row so not fully controllable.



# Summary

Used eigenvalue/vector decompositions to introduce concepts of controllability/reachability which means the ability to reach any desired value of  $x(t)$  by judicious choice of  $u(t)$ .

$$\dot{x} = Ax + Bu$$

$$A = W\Lambda V$$

- Shown that full controllability requires the matrix  $VB$  to have no zero rows.
- Not discussed non-simple Jordan forms/systems with repeated eigenvalues.



Anthony Rossiter  
Department of Automatic Control and  
Systems Engineering  
University of Sheffield  
[www.shef.ac.uk/acse](http://www.shef.ac.uk/acse)

© 2016 University of Sheffield

This work is licensed under the Creative Commons Attribution 2.0 UK: England & Wales Licence. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/2.0/uk/> or send a letter to: Creative Commons, 171 Second Street, Suite 300, San Francisco, California 94105, USA.



It should be noted that some of the materials contained within this resource are subject to third party rights and any copyright notices must remain with these materials in the event of reuse or repurposing.

If there are third party images within the resource please do not remove or alter any of the copyright notices or website details shown below the image.

*(Please list details of the third party rights contained within this work.*

*If you include your institutions logo on the cover please include reference to the fact that it is a trade mark and all copyright in that image is reserved.)*