



State-space analysis 3 controllability continued

J A Rossiter

Introduction

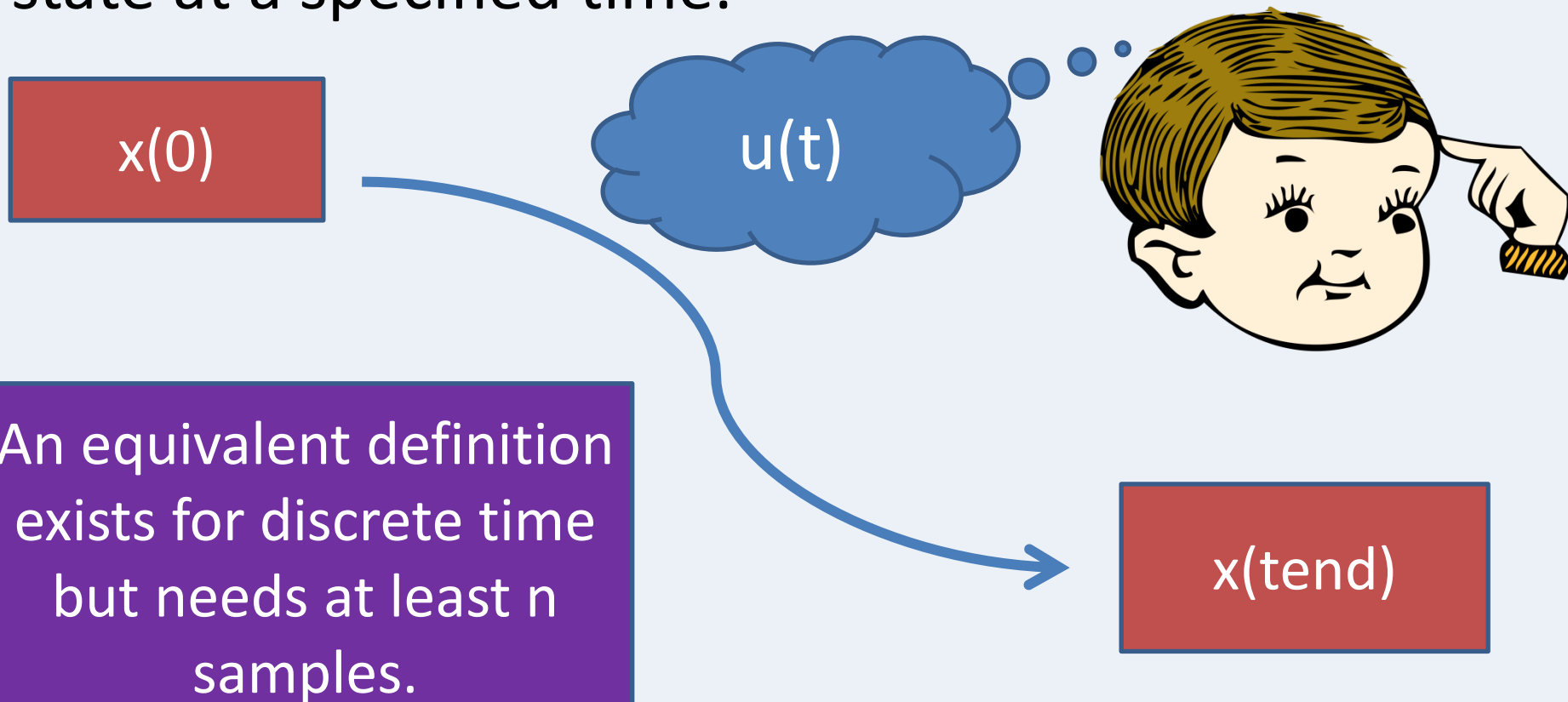
- The previous video introduced the concept of controllability, that is the ability to achieve any desired $x(t)$ by suitable choice of $u(t)$.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- The introduction used eigenvalue/vector decompositions, but in general these are hard to compute.
- This video introduces a simpler test, although the background proof is less straightforward.

Controllability/reachability

A system is controllable if it is possible to determine an appropriate input signal, for any initial state, which will achieve a specified final state at a specified time.



Remark on Controllability

A system given in control canonical form is always fully controllable.

$$\frac{d}{dt} [z] = \underbrace{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_A z + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_B u;$$

$$y = \underbrace{\begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}}_C z$$

Controllability matrix

A convenient test of controllability exists which does not necessitate the use of eigenvalue/vector decompositions.

- This reduces to testing the rank of a ‘so-called’ controllability matrix.

$$M_c = [B, AB, A^2 B, \dots, A^{n-1} B]$$

- If M_c is full rank, the system is controllable.

Here n is the dimension of the A matrix (and desired rank).

Rank deficiency equals number of uncontrollable modes.

Derivation of controllability matrix

The theoretical justification for this involves some analysis which is beyond the remit of this series but can be found in advanced text books.

It utilises the observation that one can always find coefficients γ_i such that:

$$e^{A(t-\tau)} = \sum_{i=0, \dots, n-1} \gamma_i A^i$$

$$x = \int_0^t e^{A(t-\tau)} B u d\tau \equiv \sum_{i=0, \dots, n-1} \int \gamma_i (t-\tau) A^i B u(\tau) d\tau$$

Derivation of controllability matrix

A simple rearrangement gives:

$$x = \sum_{i=0, \dots, n-1} \int \gamma_i(t - \tau) A^i B u(\tau) d\tau$$

$$\rho_i = \int \gamma_i(t - \tau) u(\tau) d\tau$$

$$x(t) = \sum_{i=0, \dots, n-1} A^i B \rho_i(t) = \underbrace{[B, AB, \dots, A^{n-1} B]}_{M_c} \begin{bmatrix} \rho_1(t) \\ \vdots \\ \rho_n(t) \end{bmatrix}$$

A long as parameters ρ_i , which depend upon $u(t)$, can be chosen freely/independently of each other, and M_c is full rank, then any $x(t)$ can be achieved.

Remark

These formulae are not used to determine $u(t)$ as that would be difficult, but rather to establish that a $u(t)$ exists.

More straightforward mechanisms exist for finding a suitable $u(t)$ which are based on feedback and thus involve negligible computation/algebra.

Example 1

Is the following system controllable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The technique is to define the controllability matrix and check it is full rank.

$$M_c = [B, AB, A^2B, \dots, A^{n-1}B]$$

$$M_c = [B, AB] = \begin{bmatrix} -2 & 4 \\ 3 & -2 \end{bmatrix}$$

Clearly full rank so controllable.

Example 2

Is the following system controllable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The technique is to define the controllability matrix and check it is full rank.

$$M_c = [B, AB, A^2B]$$

$$M_c = \begin{bmatrix} 1 & 4 & 16 \\ 2 & 5 & 21 \\ 0 & 1 & 5 \end{bmatrix}; \quad |M_c| \neq 0$$

Clearly full rank so system fully controllable.

Example 3

Is the following system controllable?

$$A = \begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.4 & -0.8 & 2.6 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The technique is to define the controllability matrix and check it is full rank.

$$M_c = [B, AB, A^2 B]$$

$$M_c = \begin{bmatrix} 1 & 2.4 & 6 \\ 1 & 2.4 & 6 \\ 0 & -1.2 & -6 \end{bmatrix}; \quad |M_c| = 0$$

Clearly rank deficient so
system not fully
controllable.

Example 4

A system on control canonical form is always fully controllable.

$$A = \begin{bmatrix} a & b & \dots & e & f \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The proof follows by showing that the matrix M_c is upper diagonal with ones on the diagonal and therefore must be full rank.

$$M_c = [B, AB, A^2 B, \dots]$$

Example 4 - continued

$$A = \begin{bmatrix} a & b & \cdots & e & f \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; AB = \begin{bmatrix} a \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

$$M_c = [B, AB, A^2B, \dots]$$

$$A(AB) = \begin{bmatrix} a^2 + b \\ a \\ 1 \\ \vdots \\ 0 \end{bmatrix}; A(A^2B) = \begin{bmatrix} a^3 + 2ab + c \\ a^2 + b \\ a \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Clear that M_c is upper triangular and so full rank.

Example 5

Is the following system controllable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 4 & 5 & -1 & 7 \\ 0.4 & 1.1 & 0 & 0 \\ 0 & 2 & -0.8 & 6 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ -1.2 \\ 3 \end{bmatrix}$$

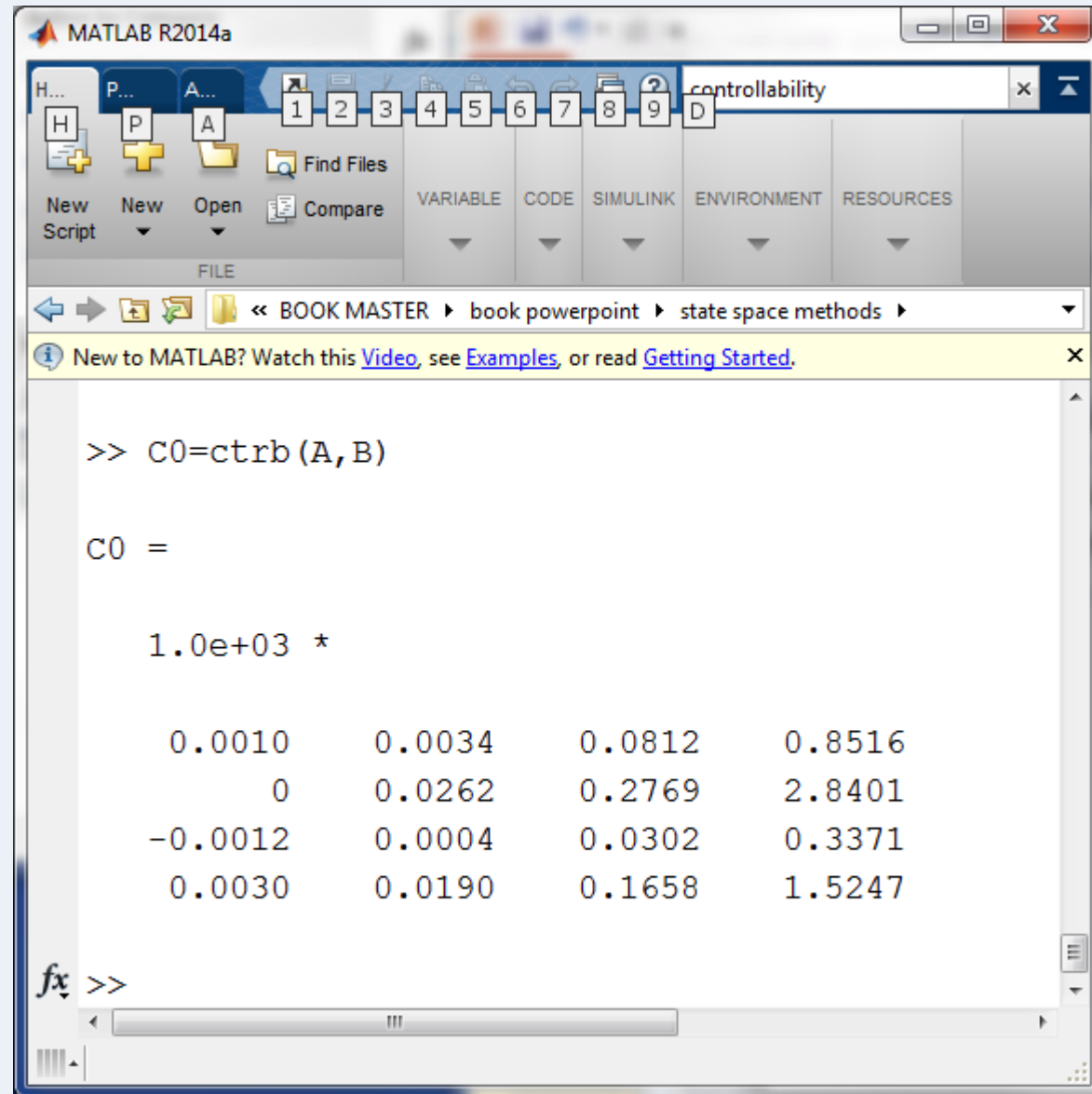
This is too tedious to do by hand so we revert to MATLAB.

$$M_c = [B, AB, A^2B, A^3B]$$

Full rank so
controllable

MATLAB shortcut

MATLAB has a built in function to find the matrix M_c – this is `ctrb.m` and its use is obvious.



The image shows a screenshot of the MATLAB R2014a interface. The command window displays the following output:

```
>> C0=ctrb(A,B)

C0 =

    1.0e+03 *

    0.0010    0.0034    0.0812    0.8516
         0    0.0262    0.2769    2.8401
   -0.0012    0.0004    0.0302    0.3371
    0.0030    0.0190    0.1658    1.5247
```

The interface also shows the MATLAB logo, the version number 'MATLAB R2014a', and a toolbar with various icons. A yellow banner at the top of the command window reads: 'New to MATLAB? Watch this [Video](#), see [Examples](#), or read [Getting Started](#).'

REMARK

If a system is uncontrollable, then the associated transfer function will have a cancelling pole/zero pair as the input is unable to excite at least one of the modes so this mode cannot appear in the IO relationship.

We will use example 3 to demonstrate this.

$$A = \begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.4 & -0.8 & 2.6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$(sI - A)^{-1} B = \frac{\begin{bmatrix} (s-1)(s-2.6) \\ (s-1)(s-2.6) \\ -1.2(s-1) \end{bmatrix}}{(s-1)(s-2)(s-3)}$$

Hence system is in non-minimal form.

Summary

Introduced concept of the controllability matrix

1. State that a system is controllable if the matrix M_c is full rank.

$$M_c = [B, AB, A^2B, \dots, A^{n-1}B]$$

2. Given some numerical examples and showed that, if in control canonical form, a state space matrix is always fully controllable.
3. In general computation of M_c is tedious so can be done on a computer.



Anthony Rossiter
Department of Automatic Control and
Systems Engineering
University of Sheffield
www.shef.ac.uk/acse

© 2016 University of Sheffield

This work is licensed under the Creative Commons Attribution 2.0 UK: England & Wales Licence. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/2.0/uk/> or send a letter to: Creative Commons, 171 Second Street, Suite 300, San Francisco, California 94105, USA.



It should be noted that some of the materials contained within this resource are subject to third party rights and any copyright notices must remain with these materials in the event of reuse or repurposing.

If there are third party images within the resource please do not remove or alter any of the copyright notices or website details shown below the image.

(Please list details of the third party rights contained within this work.

If you include your institutions logo on the cover please include reference to the fact that it is a trade mark and all copyright in that image is reserved.)