State-space analysis 3 controllability continued

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Introduction

- The previous video introduced the concept of controllability, that is the ability to achieve any desired x(t) by suitable choice of u(t).

$$\boxed{\dot{x} = Ax + Bu; \quad y = Cx + Du}$$

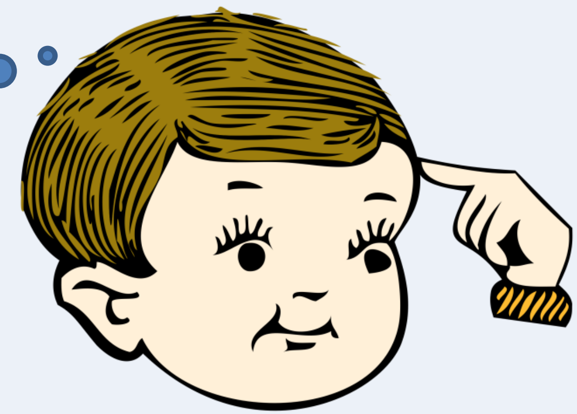
- The introduction used eigenvalue/vector decompositions, but in general these are hard to compute.
- This video introduces a simpler test, although the background proof is less straightforward.

Controllability/reachability

A system is controllable if it is possible to determine an appropriate input signal, for any initial state, which will achieve a specified final state at a specified time.

x(0)

u(t)



x(tend)

An equivalent definition exists for discrete time but needs at least n samples.

Remark on Controllability

A system given in control canonical form is always fully controllable.

$$\frac{d}{dt}[z] = \underbrace{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u;$$

$$y = \underbrace{\begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}}_{C} z$$

Controllability matrix

A convenient test of controllability exists which does not necessitate the use of eigenvalue/vector decompositions.

- This reduces to testing the rank of a ‘so-called’ controllability matrix.

$$\boxed{M_c = [B, AB, A^2B, \cdots, A^{n-1}B]}$$

- If M_c is full rank, the system is controllable.

Here n is the dimension of the A matrix (and desired rank).

Rank deficiency equals number of uncontrollable modes.

Derivation of controllability matrix

The theoretical justification for this involves some analysis which is beyond the remit of this series but can be found in advanced text books.

It utilises the observation that one can always find coefficients γ_i such that:

$$\boxed{e^{A(t-\tau)} = \sum_{i=0,\dots,n-1} \gamma_i A^i}$$

$$\boxed{x = \int_0^t e^{A(t-\tau)} Bu d\tau \equiv \sum_{i=0,\dots,n-1} \int \gamma_i (t-\tau) A^i Bu(\tau) d\tau}$$

Derivation of controllability matrix

A simple rearrangement gives:

$$x = \sum_{i=0,\ldots,n-1} \int \gamma_i(t-\tau) A^i B u(\tau) d\tau$$

$$\rho_i = \int \gamma_i(t-\tau) u(\tau) d\tau$$

$$x(t) = \sum_{i=0,\ldots,n-1} A^i B \rho_i(t) = \underbrace{[B, AB, \cdots A^{n-1}B]}_{M_c} \begin{bmatrix} \rho_1(t) \\ \vdots \\ \rho_n(t) \end{bmatrix}$$

A long as parameters ρ_i, which depend upon u(t), can be chosen freely/independently of each other, and M_c is full rank, then any x(t) can be achieved.

Remark

These formulae are not used to determine u(t) as that would be difficult, but rather to establish that a u(t) exists.

More straightforward mechanisms exist for finding a suitable u(t) which are based on feedback and thus involve negligible computation/algebra.

Example 1

Is the following system controllable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The technique is to define the controllability matrix and check it is full rank.

$$M_c = [B, AB, A^2B, \cdots, A^{n-1}B]$$

$$M_c = [B, AB] = \begin{bmatrix} -2 & 4 \\ 3 & -2 \end{bmatrix}$$

Clearly full rank so controllable.

Example 2

Is the following system controllable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The technique is to define the controllability matrix and check it is full rank.

$$M_c = [B, AB, A^2B]$$

$$M_c = \begin{bmatrix} 1 & 4 & 16 \\ 2 & 5 & 21 \\ 0 & 1 & 5 \end{bmatrix}; \quad |M_c| \neq 0$$

Clearly full rank so system fully controllable.

Example 3

Is the following system controllable?

$$A=\begin{bmatrix}1.8 & 0.6 & -0.2\\ 0.8 & 1.6 & -0.2\\ -0.4 & -0.8 & 2.6\end{bmatrix};\quad B=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

The technique is to define the controllability matrix and check it is full rank.

$$M_c=[B, AB, A^2B]$$

$$M_c=\begin{bmatrix}1 & 2.4 & 6\\ 1 & 2.4 & 6\\ 0 & -1.2 & -6\end{bmatrix};\quad |M_c|=0$$

Clearly rank deficient so system not fully controllable.

Example 4

A system on control canonical form is always fully controllable.

$$A = \begin{bmatrix} a & b & \cdots & e & f \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The proof follows by showing that the matrix M_c is upper diagonal with ones on the diagonal and therefore must be full rank.

$$\boxed{M_c = [B, AB, A^2B, \ldots]}$$

Example 4 - continued

$$A = \begin{bmatrix} a & b & \cdots & e & f \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad AB = \begin{bmatrix} a \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

$$M_c = [B, AB, A^2B, \ldots]$$

$$A(AB) = \begin{bmatrix} a^2 + b \\ a \\ 1 \\ \vdots \\ 0 \end{bmatrix}; \quad A(A^2B) = \begin{bmatrix} a^3 + 2ab + c \\ a^2 + b \\ a \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Clear that M_c is upper triangular and so full rank.

Example 5

Is the following system controllable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 4 & 5 & -1 & 7 \\ 0.4 & 1.1 & 0 & 0 \\ 0 & 2 & -0.8 & 6 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ -1.2 \\ 3 \end{bmatrix}$$

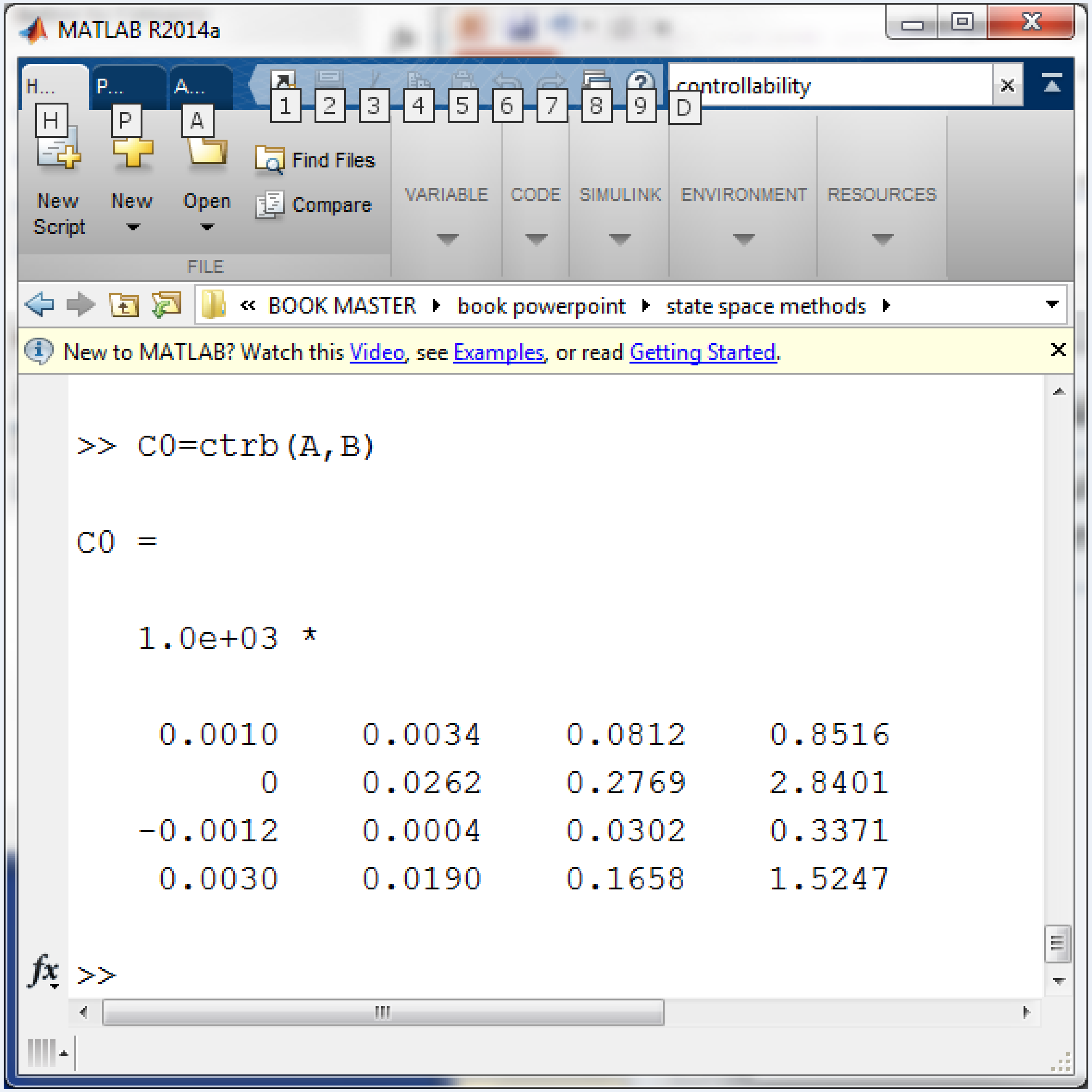
This is too tedious to do by hand so we revert to MATLAB.

$$M_c = [B, AB, A^2B, A^3B]$$

Full rank so controllable

MATLAB shortcut

MATLAB has a built in function to find the matrix M_c – this is ctrb.m and its use is obvious.



REMARK

If a system is uncontrollable, then the associated transfer function will have a cancelling pole/zero pair as the input is unable to excite at least one of the modes so this mode cannot appear in the IO relationship.

We will use example 3 to demonstrate this.

$$A=\begin{bmatrix}1.8 & 0.6 & -0.2\\ 0.8 & 1.6 & -0.2\\ -0.4 & -0.8 & 2.6\end{bmatrix}$$

$$B=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$(sI-A)^{-1}B=\frac{\begin{bmatrix}(s-1)(s-2.6)\\ (s-1)(s-2.6)\\ -1.2(s-1)\end{bmatrix}}{(s-1)(s-2)(s-3)}$$

Hence system is in non-minimal form.

Summary

Introduced concept of the controllability matrix

1. State that a system is controllable if the matrix M_c is full rank.

$$M_c = [B, AB, A^2B, \cdots, A^{n-1}B]$$

2. Given some numerical examples and showed that, if in control canonical form, a state space matrix is always fully controllable.
3. In general computation of M_c is tedious so can be done on a computer.

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