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State-space analysis 4 controllability discrete systems

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Introduction

- The last two videos considered controllability for continuous time state space models.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- This video considers to what extent concepts and tests for controllability differ for discrete state space models.

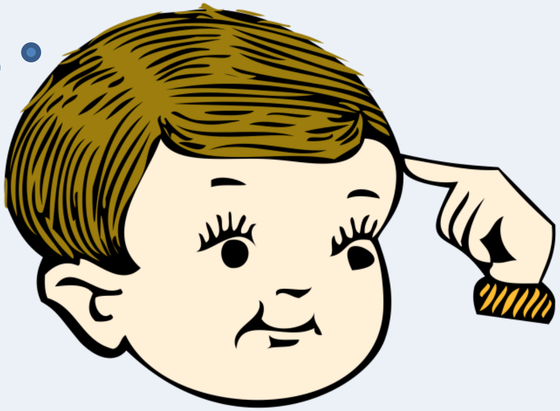
$$x_{k+1} = Ax_k + Bu_k$$

Controllability/reachability

A system is controllable if it is possible to determine an appropriate input signal, for any initial state, which will achieve a specified final state at a specified time.

x(0)

u(k)



x(n)

In discrete time one chooses values of u(k) at each sample.

Remark

1. In continuous time, the end time at which the desired state was to be achieved did not need to be defined or constrained in any way.
2. This is not the case with discrete time because the degrees of freedom are much more constrained.
3. Specifically, the input can only be changed at each sampling instant rather than continuously.
4. To meet n constraints (that is to specify an n-dimensional state vector), will require at least n d.o.f. in general, and thus at least n samples.

Prediction with discrete models

It is straightforward to use recursion on a nominal state space model to find the state n-steps ahead.

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
x_{k+2} &= Ax_{k+1} + Bu_{k+1} \\
x_{k+3} &= Ax_{k+2} + Bu_{k+2} \\
&\vdots \\
x_{k+n} &= Ax_{k+n-1} + Bu_{k+n-1}
\end{aligned}
$$

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
x_{k+2} &= A^2 x_k + Bu_{k+1} + ABu_k \\
x_{k+3} &= A^3 x_k + Bu_{k+2} + ABu_{k+1} + A^2 Bu_k \\
&\vdots \\
x_{k+n} &= A^n x_{k+n-1} + \sum_{i=0}^{n-1} A^i Bu_{k+n-i-1}
\end{aligned}
$$

For convenience express in matrix form

N-step ahead state prediction

The final prediction equation is summarised as follows.

$$x_{k+n} = A^n x_{k+n-1} + \sum_{i=0}^{n-1} A^i B u_{k+n-i-1}$$

$$x_{k+n} = A^n x_{k+n-1} + [B, AB, \cdots, A^{n-1}B] \begin{bmatrix} u_{k+n-1} \\ u_{k+n-2} \\ \vdots \\ u_k \end{bmatrix}$$

Controllability matrix M_c

If M_c is full rank, there are enough d.o.f. to place x_{k+n} wherever we like!

Summary

The same controllability matrix as defined for continuous time also works for discrete time.

A system is fully controllable if the controllability matrix is full rank.

$$x_{k+1} = Ax_k + Bu_k$$

$$M_c = [B, AB, A^2B, \cdots, A^{n-1}B]$$

Eigenvalue/vector decomposition

Viewers may be interested in whether there are also analogies with the insights derived from the eigenvector/value decomposition.

In the continuous time case the result was summarised as:

$$\dot{x} = Ax + Bu \qquad A = W \Lambda V$$

FULLY CONTROLLABLE IF VB HAS NO ZERO ROWS.

Using eigenvalue/vector decomposition

This decomposition is useful for identifying whether there are regions in the state space which cannot be reached, from an arbitrary start point.

Without out loss of generality, we will assume the initial condition is zero!

$$x(n) = \sum_{i=0}^{n-1} A^i B u_{k+n-i-1}; \quad A^i = W\Lambda^i V$$

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$$

W is full rank.

Using eigenvalue/vector decomposition

It is clear that x(n) has the following structure:

$$x(n) = \sum_{i=0}^{n-1} A^i B u_{k+n-i-1} = \sum_{i=0}^{n-1} W\Lambda^i V B u_{k+n-i-1}$$

$$\sum_{i=0}^{n-1} W\Lambda^i V B u_{k+n-i-1} = \left(\sum_{i=0}^{n-1} \left(\sum_{j} w_j \lambda_j^i v_j^T \right) B u_{k+n-i} \right)$$

Next separate each mode for convenience.

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Using eigenvalue/vector decomposition

Separating each mode gives:

$$x(n)=\sum_{i=0}^{n-1} w_1\lambda_1^i v_1^T Bu_{k+n-i}+\sum_{i=0}^{n-1} w_2\lambda_2^i v_2^T Bu_{k+n-i}+\cdots$$

$$x(n)=\sum_j w_j\alpha_j \qquad \alpha_j=\left(\sum_{i=0}^{n-1}\lambda_j^i v_j^T Bu_{k+n-i}\right)$$

It is clear that to choose x(n) arbitrarily 2 conditions are required:

- $\alpha_j \neq 0$, and indeed can be specified as required.
- W is full rank (**already known**).

Using eigenvalue/vector decomposition

We require that α_i can be chosen freely.

$$\boxed{\alpha_j = \left(\sum_{i=0}^{n-1} \lambda_j^i v_j^T B u_{k+n-i} \right)} \qquad \boxed{\beta_j^T = v_j^T B}$$

Assuming that input u(k) is a free variable, then α_j can be chosen freely iff β_j is non-zero.

It is clear that $\beta_j=0 \rightarrow \alpha_j=0$, so a final state with a component in the corresponding eigenvector direction cannot be reached!

NOTE: Same result as for continuous time case!

Eigenvalue/vector decomposition

In both the continuous time case and discrete case, controllability can be tested using the eigenvalue/vector decomposition.

$$\dot{x} = Ax + Bu$$

$$x_{k+1} = Ax_k + Bu_k$$

$$A = W\Lambda V$$

FULLY CONTROLLABLE IF VB HAS NO ZERO ROWS.

Summary

The tests for controllability are the same for both the continuous and discrete cases.

1. Either form the controllability matrix and ensure it is full rank.

$$\boxed{M_c = [B, AB, A^2B, \cdots, A^{n-1}B]}$$

2. Or, do the eigenvalue/vector decomposition and ensure that VB has no zero rows.

The former of these is often easier to use.

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