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State-space analysis 5 controllability worked examples

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Introduction

- It has been shown that there are two simple controllability tests for state space models.
- These tests are the same for both continuous and discrete models (with n states).

$$\dot{x} = Ax + Bu; \quad x_{k+1} = Ax_k + Bu_k$$

1. The controllability matrix must be full rank:

$$M_c = [B, AB, A^2B, \cdots, A^{n-1}B]$$

2. The matrix VB must have no zero rows where:

$$A = W\Lambda V$$

Remark

In practice an eigenvalue/vector decomposition is non-trivial and not a paper and pen exercise.

Consequently these examples will only use the controllability matrix.

$$M_c = [B, AB, A^2B, \cdots, A^{n-1}B]$$

1. However, it is reiterated, that in the vast majority of cases you are advised to use a computer as the computations are tedious.
2. Hand calculations are useful to gain insight and understanding.

Example 1

$$A=\begin{bmatrix}1 & 2\\ 5 & 0\end{bmatrix};\quad B=\begin{bmatrix}0\\ 2\end{bmatrix}$$

$$M_c=[B, AB]$$

$$AB=\begin{bmatrix}4\\ 0\end{bmatrix}$$

$$M_c=\begin{bmatrix}0 & 4\\ 2 & 0\end{bmatrix}$$

FULL RANK

Example 2 (controllable canonical form)

$$A=\begin{bmatrix}1 & 3 & 3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix};\quad B=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$M_c=[B, AB, A^2B]$$

$$AB=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$A^2B=A(AB)=\begin{bmatrix}4\\1\\1\end{bmatrix}$$

$$M_c=\begin{bmatrix}1 & 1 & 4\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

FULL RANK

Example 3 (alternative control canonical form)

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 1 & 4\end{bmatrix}; \quad B=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$M_c=[B, AB, A^2B]$$

$$AB=\begin{bmatrix}0\\1\\4\end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix}1\\4\\17\end{bmatrix}$$

$$M_c=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 4\\ 1 & 4 & 17\end{bmatrix}$$

FULL RANK

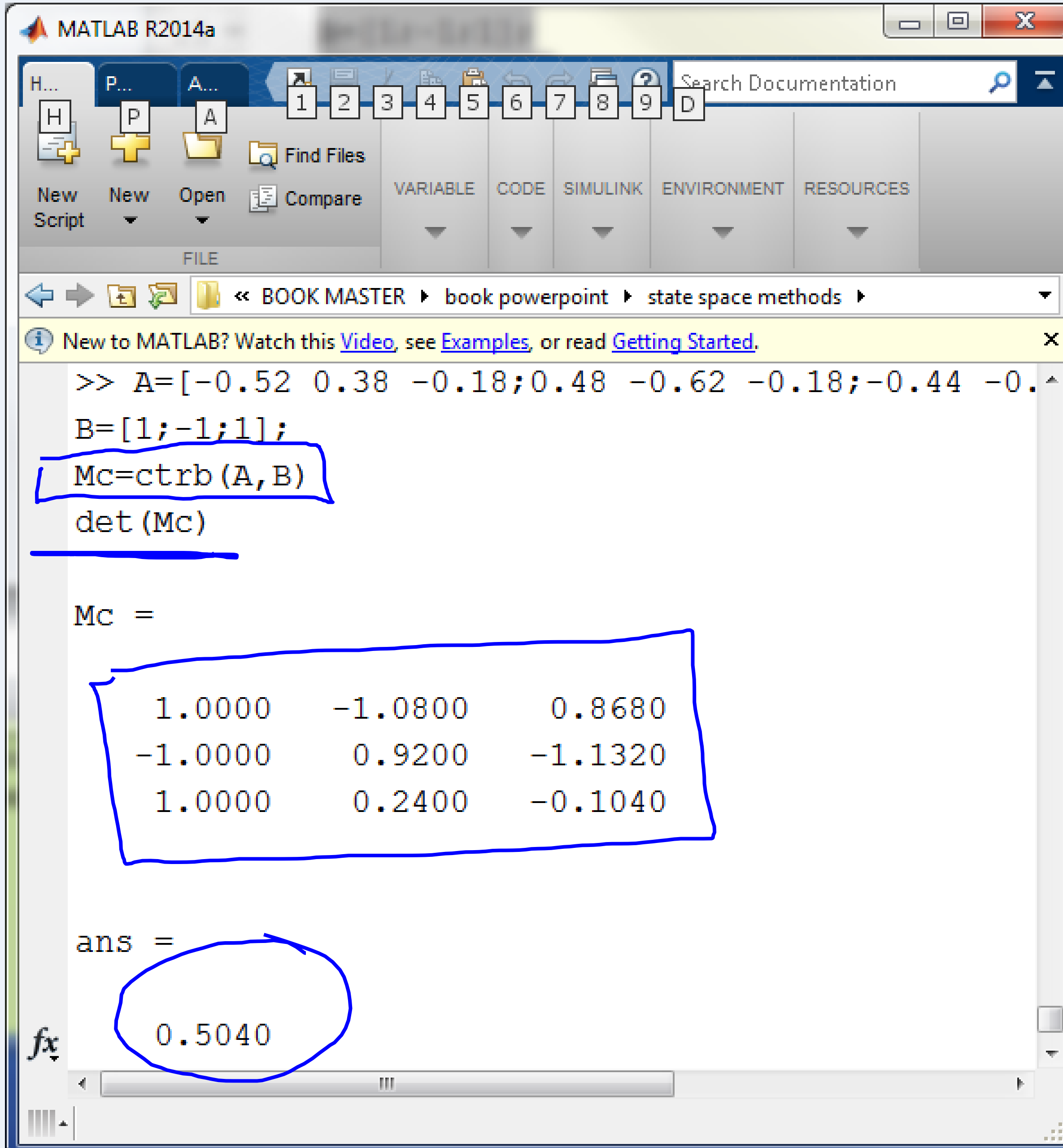
Example 4 (use MATLAB)

$$A = \begin{bmatrix} -0.52 & 0.38 & -0.18 \\ 0.48 & -0.62 & -0.18 \\ -0.44 & -0.64 & 0.04 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$M_c = [B, AB, A^2B]$$

RANK



Example 5

Find ‘k’ such that the following example is uncontrollable and hence show that the associated transfer function has a cancelling pole/zero pair.

$$A = \begin{bmatrix} -1 & 0.4 \\ k & -1.2 \end{bmatrix}; \quad B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$M_c = [B, AB]$$

$$M_c = \begin{bmatrix} -1 & 1.8 \\ 2 & -k-2.4 \end{bmatrix}$$

$$|M_c$$

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$$G = C(sI - A)^{-1}B$$

$$= C\begin{bmatrix} s+1 & -0.4 \\ -1.2 & s+1.2 \end{bmatrix}^{-1} B = \frac{C\begin{bmatrix} s+1.2 & 0.4 \\ 1.2 & s+1 \end{bmatrix} B}{s^2 + 2.2s + 0.72}$$

$$G(s) = \frac{C\begin{bmatrix} -(s+0.4) \\ 2(\cancel{s+0.4}) \end{bmatrix}}{(s+1.8)(\cancel{s+0.4})}$$

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