



# State-space analysis 5

## controllability worked examples

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# Introduction

- It has been shown that there are two simple controllability tests for state space models.
- These tests are the same for both continuous and discrete models (with  $n$  states).

$$\dot{x} = Ax + Bu; \quad x_{k+1} = Ax_k + Bu_k$$

1. The controllability matrix must be full rank:

$$M_c = [B, AB, A^2B, \dots, A^{n-1}B]$$

2. The matrix  $VB$  must have no zero rows where:

$$A = W\Lambda V$$

# Remark

In practice an eigenvalue/vector decomposition is non-trivial and not a paper and pen exercise.

Consequently these examples will only use the controllability matrix.

$$M_c = [B, AB, A^2 B, \dots, A^{n-1} B]$$

1. However, it is reiterated, that in the vast majority of cases you are advised to use a computer as the computations are tedious.
2. Hand calculations are useful to gain insight and understanding.

# Example 1

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$M_c = [B, AB]$$

$$AB = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$$

FULL RANK  
 ,,

# Example 2 (controllable canonical form)

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$M_c = [B, AB, A^2B]$$

$$AB = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

FULL RANK

# Example 3 (alternative control canonical form)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$M_c = [B, AB, A^2B]$$

$$AB = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} 1 \\ 4 \\ 17 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 4 & 17 \end{bmatrix}$$

FULL RANK

# Example 4 (use MATLAB)

$$A = \begin{bmatrix} -0.52 & 0.38 & -0.18 \\ 0.48 & -0.62 & -0.18 \\ -0.44 & -0.64 & 0.04 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$M_c = [B, AB, A^2B]$$

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Info: New to MATLAB? Watch this [Video](#), see [Examples](#), or read [Getting Started](#).

```
>> A=[-0.52 0.38 -0.18;0.48 -0.62 -0.18;-0.44 -0.18 -0.64];
B=[1;-1;1];
Mc=ctrb(A,B)
det(Mc)
```

Mc =

1.0000	-1.0800	0.8680
-1.0000	0.9200	-1.1320
1.0000	0.2400	-0.1040

ans =

0.5040

# Example 5

Find 'k' such that the following example is uncontrollable and hence show that the associated transfer function has a cancelling pole/zero pair.

$$A = \begin{bmatrix} -1 & 0.4 \\ k & -1.2 \end{bmatrix}; \quad B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \quad C = [1 \quad 3]$$

$$M_c = [B, AB]$$

$$M_c = \begin{bmatrix} -1 & 1.08 \\ 2 & -k - 2.4 \end{bmatrix}$$

$$|M_c$$



$$\begin{aligned}
 G &= C(sI - A)^{-1}B \\
 &= C \begin{bmatrix} s+1 & -0.4 \\ -1.2 & s+1.2 \end{bmatrix}^{-1} B = C \frac{\begin{bmatrix} s+1.2 & 0.4 \\ 1.2 & s+1 \end{bmatrix} B}{s^2 + 2.2s + 0.72}
 \end{aligned}$$

$$G(s) = \frac{C \begin{bmatrix} -(s+0.4) \\ 2(s+0.4) \end{bmatrix}}{(s+1.8)(s+0.4)}$$



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