



# State-space analysis 6 observability

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# Introduction

- The previous video looked at definitions of controllability, the ability to place a state  $x(t)$  arbitrarily.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

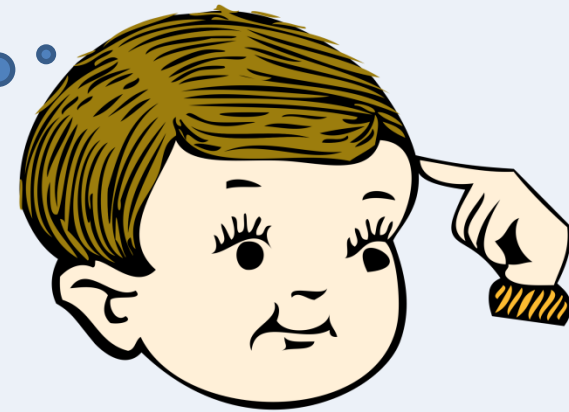
- The next focus is on observability, which is whether, **using limited measurement information**, one can identify the value  $x(t)$ .
- Clearly in general, knowledge of  $x(t)$  is useful and moreover, so is our level of confidence in the estimated value, but it is rare that this can be measured directly.

# Observability

A system is observable if it is possible to determine the value of the state  $x(t)$  from available measurement data (usually the inputs and outputs).

$y(0), u(0)$

$x(t)$



An equivalent definition exists for discrete time.

$y(t), u(t)$

# Remark on Observability

A system given in observable canonical form is always fully observable.

$$\frac{d}{dt} [z] = \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}}_A z + \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}}_B u;$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}}_C z$$

# Using eigenvalue/vector decomposition

A system is fully observable if and only if the CW matrix has no zero columns.

$$A = W\Lambda V$$

$$\begin{aligned} \dot{x} &= Ax + Bu; \\ y &= Cx + Du \end{aligned}$$

$$\begin{cases} V\dot{x} = VAx + VBu \\ y = CWVx + Du \end{cases} \Rightarrow \begin{cases} \dot{z} = \Lambda z + VBu \\ y = CWz + Du \end{cases}$$

It is clear that if CW has a zero column, then the corresponding component of z cannot be measured.

# Using eigenvalue/vector decomposition

A system is fully observable if and only if the CW matrix has no zero columns.

$$\left\{ \begin{array}{l} V\dot{x} = VAx + VBu \\ y = CWVx + Du \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \dot{z} = \Lambda z + VBu \\ y = CWz + Du \end{array} \right\}$$

$$A = W\Lambda V; \quad [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n] = CW$$

It is clear that if  $\gamma_i=0$ , then the corresponding component of  $z$ , which is the contribution of a particular eigenmode, cannot be measured.

If  $\gamma_i \neq 0$ , then there always exists enough information in  $y(t)$  to determine the underlying  $z(t)$  and therefore, by inference,  $x(t)$ !

# NUMERICAL EXAMPLES

# Example 1

Is the following system observable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

First do the eigenvalue/vector decomposition.

$$|\lambda I - A| = 0 \Rightarrow \lambda = 2, -1$$

$$(A - 2I)w_1 \Rightarrow \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A + I)w_2 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



# Example 1 continued

The eigenvector matrix is therefore given as?

$$W = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Clearly all columns are non-zero so observable.

Therefore, one can find  $CW$  as:

$$CW = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

What would happen if  $C=[1 \ 1]$ ?

# Example 2

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}; \quad C = [1 \quad 0 \quad 1]$$

First do the eigenvalue/vector decomposition.

$$|\lambda I - A| = 0 \quad \Rightarrow \quad \lambda = 4.24, -0.24, 0$$

$$(A - \lambda_1 I)w_1 \quad \Rightarrow \quad w_1 = \begin{bmatrix} -0.6 \\ -0.78 \\ -0.18 \end{bmatrix}$$

Not paper and pen computations so we revert to MATLAB to demonstrate more.

# Example 2 continued

MATLAB is used to solve:

$$A = W\Lambda V$$

$$[\gamma_1 \quad \gamma_2 \quad \dots \quad \gamma_n] = CW$$

```

>> A=[2 1 3;3 1 4;1 0 1 ];B=[1;2;0];C=[1 0 1]
[W,l]=eig(A);
C*W

ans =

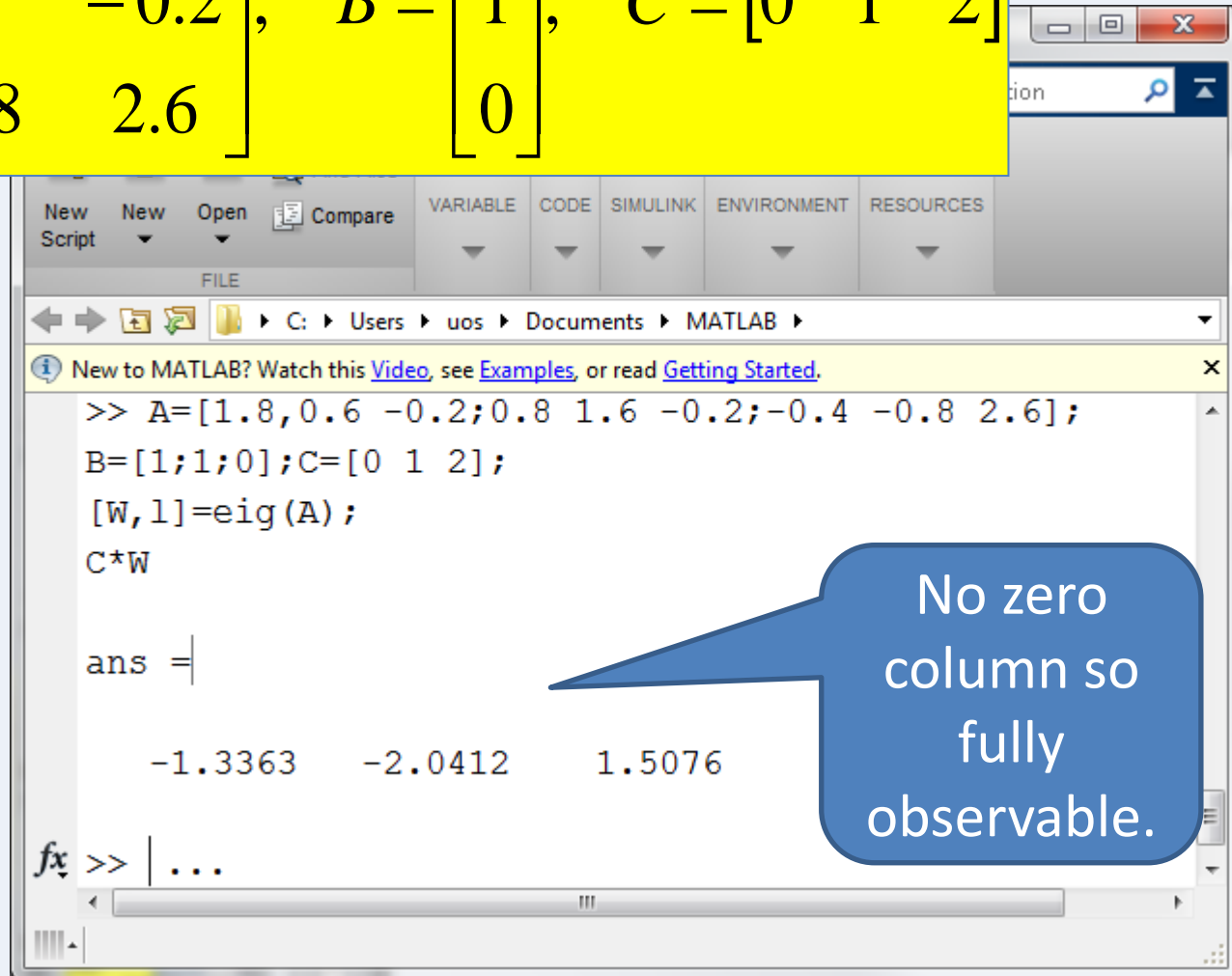
-0.7810 -0.1469 -0.0000
    
```

Zero column  
so 3<sup>rd</sup> mode  
not  
observable.

# Example 3

$$A = \begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.4 & -0.8 & 2.6 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad C = [0 \quad 1 \quad 2]$$

Using MATLAB to compute the decomposition and CW matrix gives:



```

New Script  New  Open  Compare  VARIABLE  CODE  SIMULINK  ENVIRONMENT  RESOURCES
FILE
C:\Users\uos\Documents\MATLAB
New to MATLAB? Watch this Video, see Examples, or read Getting Started.
>> A=[1.8,0.6 -0.2;0.8 1.6 -0.2;-0.4 -0.8 2.6];
B=[1;1;0];C=[0 1 2];
[W,l]=eig(A);
C*W

ans =

-1.3363    -2.0412    1.5076

fx >> | ...

```

No zero column so fully observable.

# Summary

Used eigenvalue/vector decompositions to introduce concepts of observability which means the ability to determine  $x(t)$  from the available measurements.

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$A = W\Lambda V$$

- Shown that full observability requires the matrix  $CW$  to have no zero columns.
- Not discussed non-simple Jordan forms/systems with repeated eigenvalues.



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