# State-space analysis 6 observability

J A Rossiter

Slides by Anthony Rossiter

# Introduction

- The previous video looked at definitions of controllability, the ability to place a state x(t) arbitrarily.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- The next focus is on observability, which is whether, **using limited measurement information**, one can identify the value x(t).
- Clearly in general, knowledge of x(t) is useful and moreover, so is our level of confidence in the estimated value, but it is rare that this can be measured directly.

# Observability

A system is observable if it is possible to determine the value of  the state x(t) from available measurement data (usually the inputs and outputs).

y(0), u(0)


An equivalent definition exists for discrete time.

y(t), u(t)

# Remark on Observability

A system given in observable canonical form is always fully observable.

$$\frac{d}{dt}[z] = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}}_{B} u;$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}}_{C} z$$

# Using eigenvalue/vector decomposition

A system is fully observable if and only if the CW matrix has no zero columns.

$$A = W\Lambda V$$

$$\dot{x} = Ax + Bu; \qquad y = Cx + Du$$

$$\left\{\begin{matrix} V\dot{x} = VAx + VBu \\ y = CWVx + Du \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \dot{z} = \Lambda z + VBu \\ y = CWz + Du \end{matrix}\right\}$$

It is clear that if CW has a zero column, then the corresponding component of z cannot be measured.

# Using eigenvalue/vector decomposition

A system is fully observable if and only if the CW matrix has no zero columns.

$$\begin{Bmatrix} V\dot{x} = VAx + VBu \\ y = CWVx + Du \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dot{z} = \Lambda x + VBu \\ y = CWz + Du \end{Bmatrix}$$

$$A = W\Lambda V; \quad \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix} = CW$$

It is clear that if $\Upsilon_i$=0, then the corresponding component of z, which is the contribution of a particular eigenmode, cannot be measured.

If $\Upsilon_i$≠0, then there always exists enough information in y(t) to determine the underlying z(t) and therefore, by inference, x(t)!

# NUMERICAL EXAMPLES

# Example 1

Is the following system observable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

First do the eigenvalue/vector decomposition.

$$|\lambda I - A| = 0 \quad \Rightarrow \quad \lambda = 2, -1$$

$$(A - 2I)w_1 \quad \Rightarrow \quad \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A + I)w_2 \quad \Rightarrow \quad \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

# Example 1 continued

The eigenvector matrix is therefore given as?

$$W = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Clearly all columns are non-zero so observable.

Therefore, one can find CW as:

$$CW = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

What would happen if C=[1 1]?

# Example 2

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

First do the eigenvalue/vector decomposition.

$$|\lambda I - A| = 0 \quad \Rightarrow \quad \lambda = 4.24, -0.24, 0$$

$$(A - \lambda_1 I)w_1 \quad \Rightarrow \quad w_1 = \begin{bmatrix} -0.6 \\ -0.78 \\ -0.18 \end{bmatrix}$$

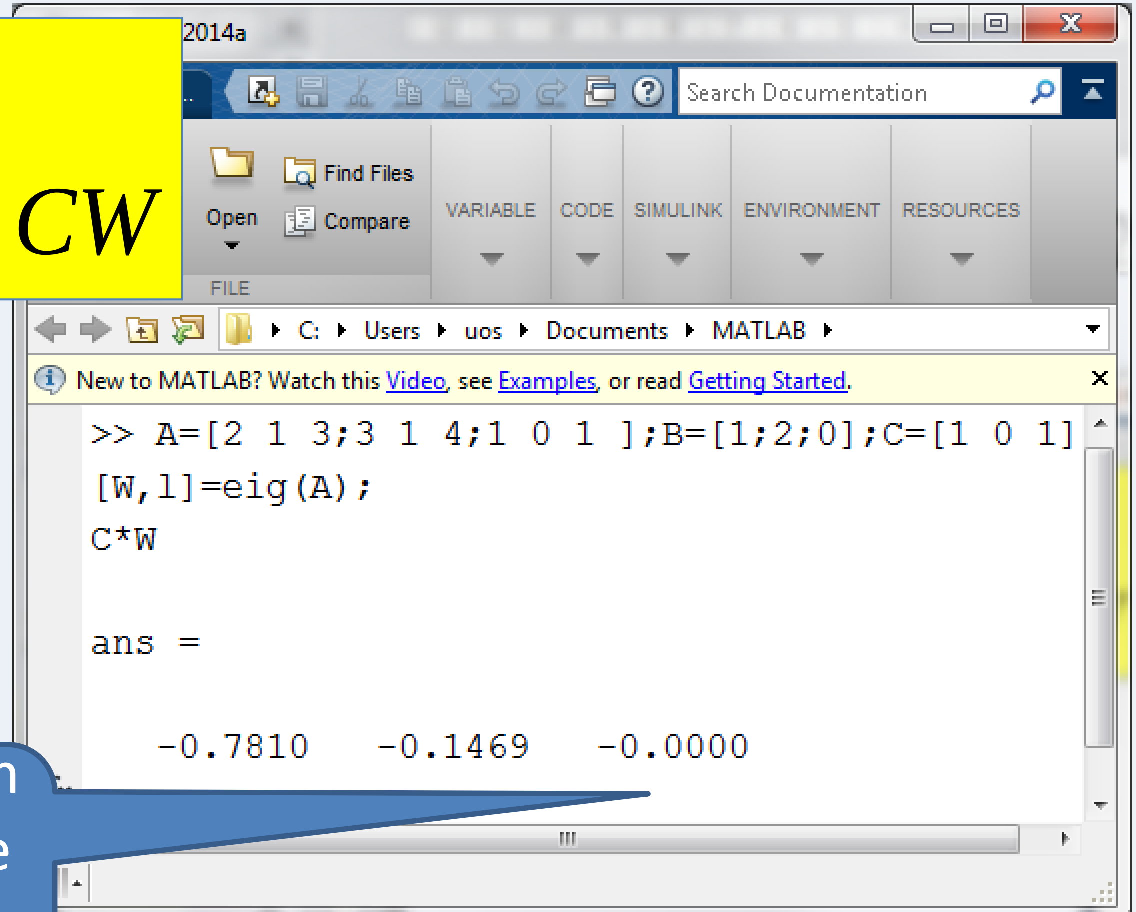
Not paper and pen computations so we revert to MATLAB to demonstrate more.

# Example 2 continued

MATLAB is used to solve:

$$A = W\Lambda V$$

$$\begin{bmatrix}\gamma_1 & \gamma_2 & \cdots & \gamma_n\end{bmatrix} = CW$$

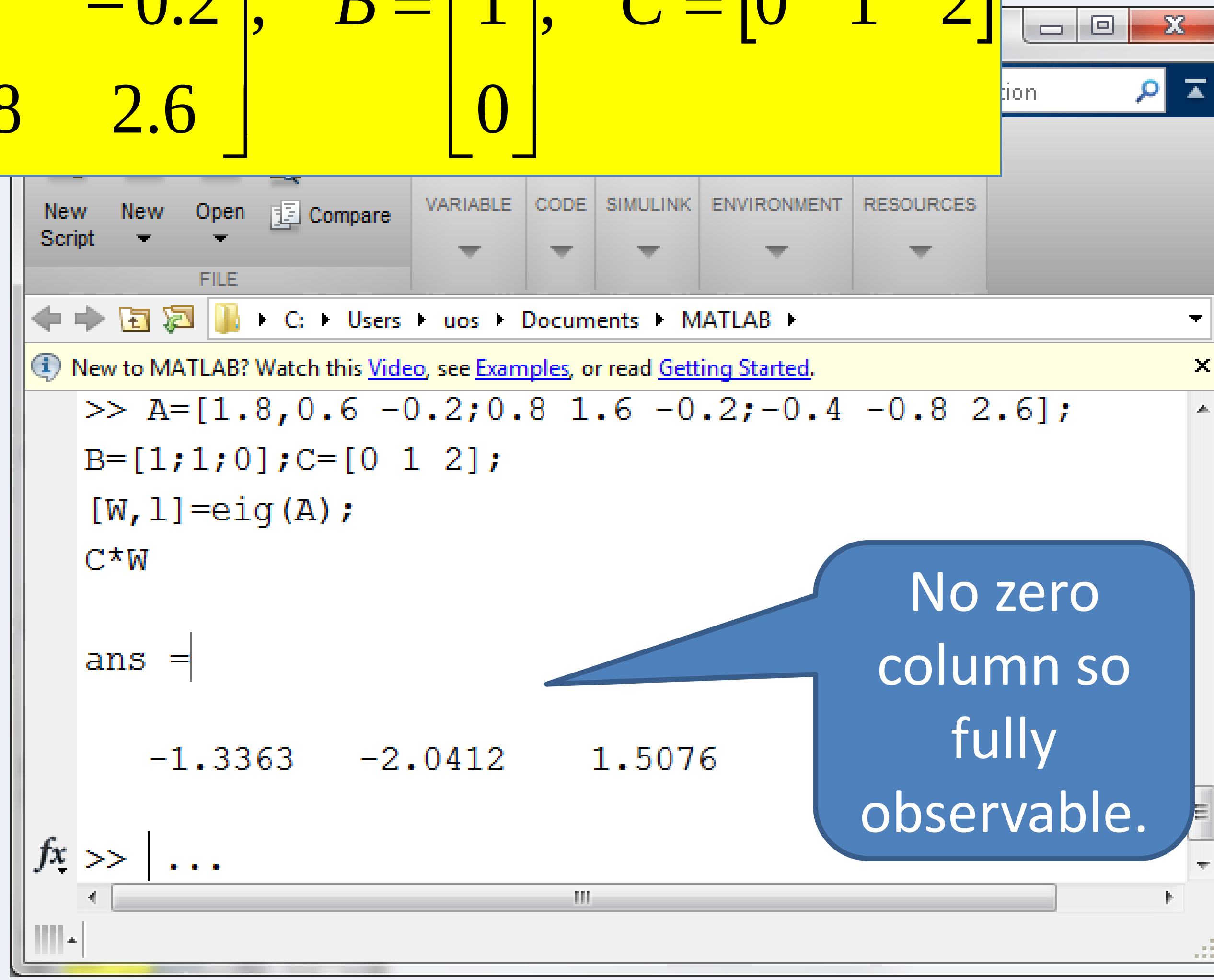



Zero column so 3rd mode not observable.

# Example 3

$$A=\begin{bmatrix}1.8 & 0.6 & -0.2\\ 0.8 & 1.6 & -0.2\\ -0.4 & -0.8 & 2.6\end{bmatrix};\quad B=\begin{bmatrix}1\\1\\0\end{bmatrix};\quad C=\begin{bmatrix}0 & 1 & 2\end{bmatrix}$$

Using MATLAB to compute the decomposition and CW matrix gives:




No zero column so fully observable.

# Summary

Used eigenvalue/vector decompositions to introduce concepts of observability which means the ability to determine x(t) from the available measurements.

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$A = W\Lambda V$$

- Shown that full observability requires the matrix CW to have no zero columns.
- Not discussed non-simple Jordan forms/systems with repeated eigenvalues.

Anthony Rossiter
Department of Automatic Control and Systems Engineering
University of Sheffield
www.shef.ac.uk/acse