State-space analysis 7 observability continued

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Introduction

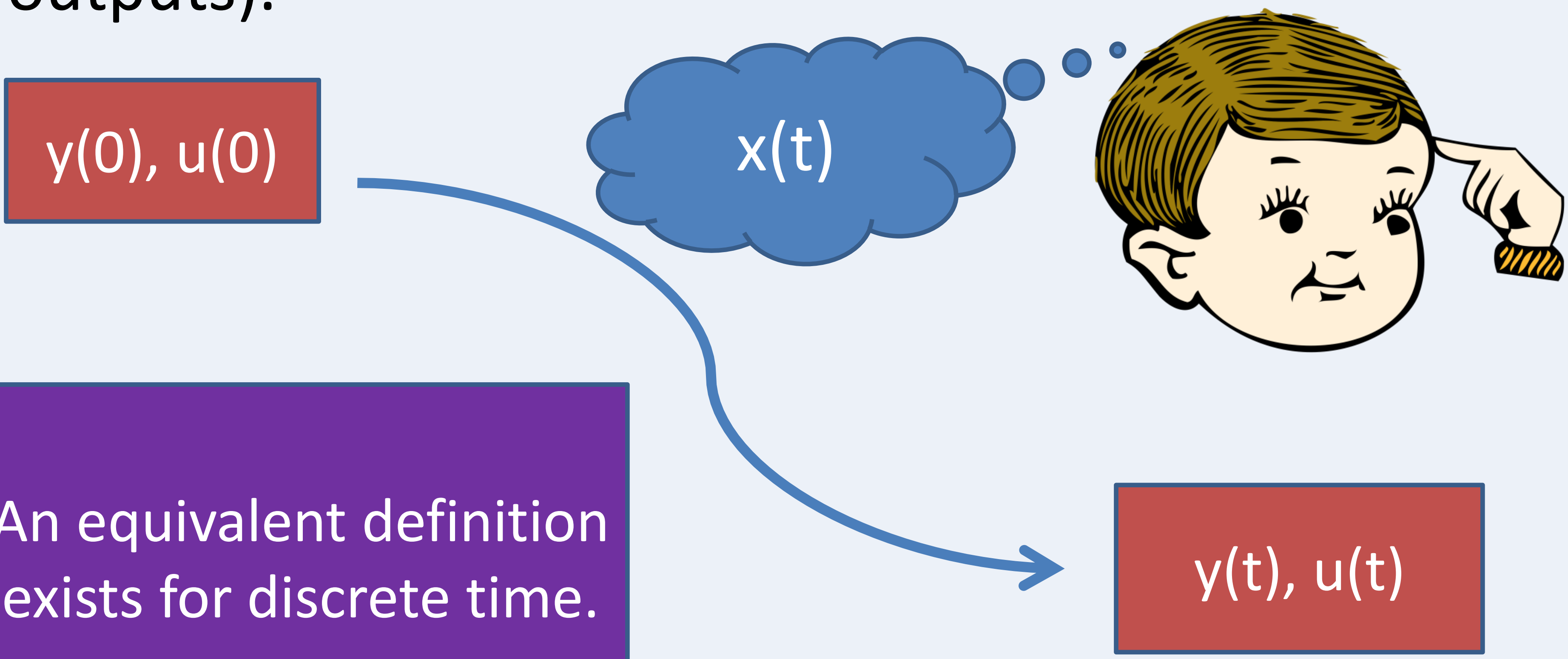
- The previous video looked at definitions of observability using eigenvalue/vector decompositions.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- Observability determines whether, **using limited measurement information**, one can identify the value x(t).
- This video gives a test which is far easier to implement than using decompositions.

Observability

A system is observable if it is possible to determine the value of the state x(t) from available measurement data (usually the inputs and outputs).



An equivalent definition exists for discrete time.

Using eigenvalue/vector decomposition

A system is fully observable if and only if the CW matrix has no zero columns.

$$A = W\Lambda V$$

$$\dot{x} = Ax + Bu; \qquad y = Cx + Du$$

$$\begin{Bmatrix} V\dot{x} = VAx + VBu \\ y = CWVx + Du \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dot{z} = \Lambda x + VBu \\ y = CWz + Du \end{Bmatrix}$$

It is clear that if CW has a zero column, then the corresponding component of z cannot be measured.

Remark on Observability

A system given in observable canonical form is always fully observable.

$$\frac{d}{dt}[z]=\underbrace{\begin{bmatrix}0 & 0 & \cdots & 0 & -a_0\\ 1 & 0 & \cdots & 0 & -a_1\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & -a_{n-1}\end{bmatrix}}_{A}z+\underbrace{\begin{bmatrix}b_0\\ b_1\\ \vdots\\ b_{n-1}\end{bmatrix}}_{B}u;$$

$$y=\underbrace{\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}}_{C}z$$

Observability matrix

A convenient test of observability exists which does not necessitate the use of eigenvalue/vector decompositions.

- This reduces to testing the rank of a 'so-called' observability matrix.
- If M_o is full rank, the system is observable.

$$M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Here n is the dimension of the A matrix (and desired rank).

Rank deficiency equals number of unobservable modes.

Derivation of observability matrix

The theoretical justification for this involves some analysis which is beyond the remit of this series but can be found in advanced text books.

It utilises the observation that one can always find coefficients γ_i such that:

$$e^{A(t-\tau)} = \sum_{i=0,\ldots,n-1} \gamma_i A^i$$

Remark

These formulae are used to determine whether x(t) can be estimated.

1. This is not the same as saying how x(t) is estimated.
2. The latter topic is covered later in sections on observers.

Duality

Readers will have noticed a strong connection between the observability and controllability matrices.

$$M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$M_c = [B, AB, A^2B, \cdots, A^{n-1}B]$$

$$M_o^T = [C^T, A^TC^T, (A^T)^2C^T, \cdots, (A^T)^{n-1}C^T]$$

Observability test is equivalent to a controllability test on a dual system:

$$\dot{x} = A^Tx + C^Tu$$

NUMERICAL EXAMPLES

Example 1

Is the following system observable?

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

First form the observability matrix.

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

Clearly full rank so observable.

What would happen if C=[1 1]?

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Example 2

$$\dot{x} = Ax + Bu; \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

First form the observability matrix.

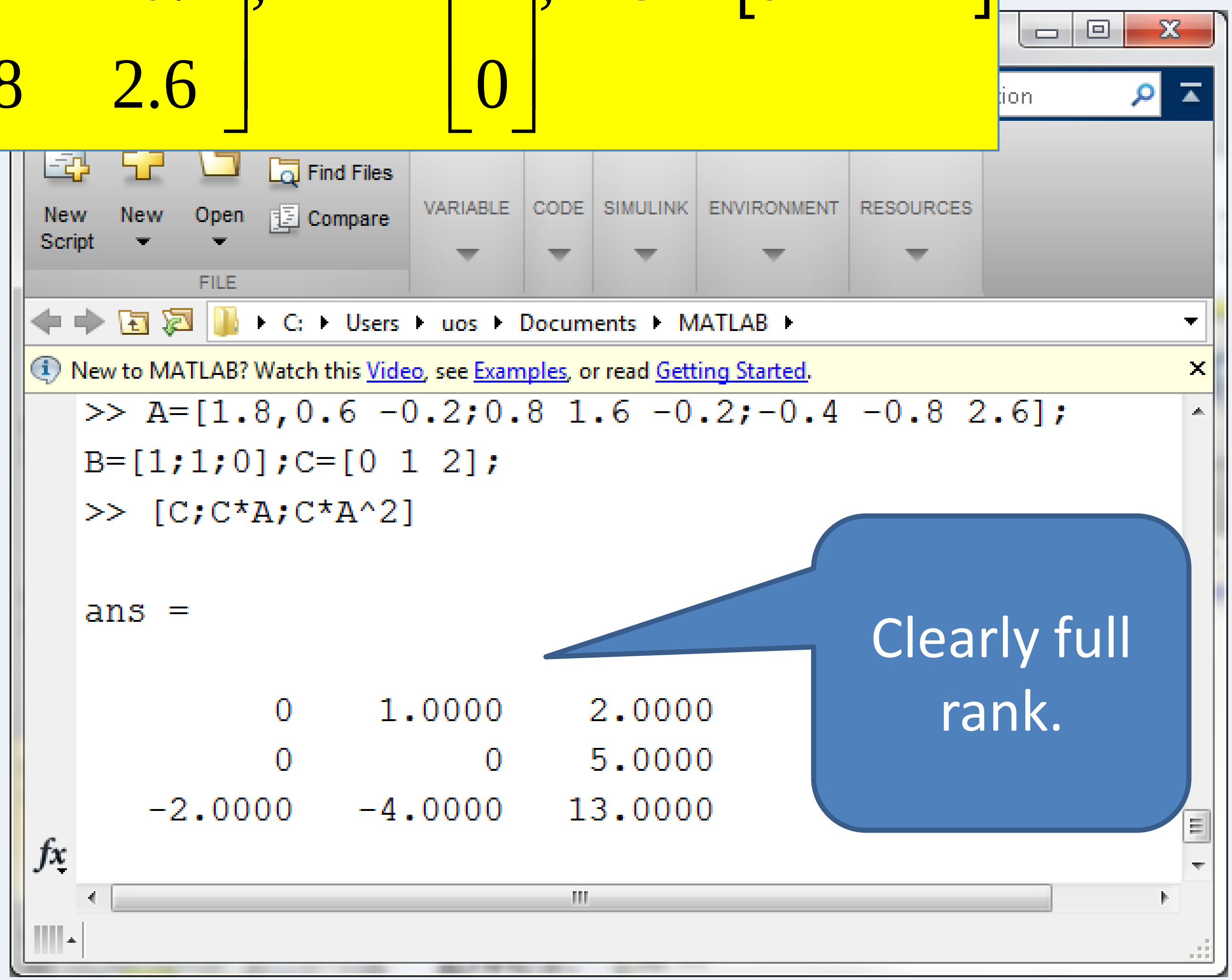
$$M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 4 \\ 13 & 4 & 17 \end{bmatrix}$$

This is clearly rank deficient.
(Last column is sum of 1st two columns)

Example 3

$$A=\begin{bmatrix}1.8 & 0.6 & -0.2\\ 0.8 & 1.6 & -0.2\\ -0.4 & -0.8 & 2.6\end{bmatrix};\quad B=\begin{bmatrix}1\\1\\0\end{bmatrix};\quad C=\begin{bmatrix}0 & 1 & 2\end{bmatrix}$$

Using MATLAB to compute the observability matrix.



Example 4 (observer canonical form)

$$A=\begin{bmatrix}0 & 0 & a\\ 1 & 0 & b\\ 0 & 1 & c\end{bmatrix}; \quad B=\begin{bmatrix}d\\ e\\ f\end{bmatrix}; \quad C=\begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

First form the observability matrix.

$$M_o=\begin{bmatrix}C\\ CA\\ \vdots\\ CA^{n-1}\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & c\\ 1 & c & b+c^2\end{bmatrix}$$

This is clearly full rank, irrespective of the values a,b,c!

Example 5 (controllable canonical form)

$$A=\begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad B=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad C=\begin{bmatrix} d & e & f \end{bmatrix}$$

First form the observability matrix.

$$M_o=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}=\begin{bmatrix} d & e & f \\ e+ad & f+bd & cd \\ f+ae+d(a^2+b) & be+d(c+ab) & c+acd \end{bmatrix}$$

It is clear that the rank can vary depending upon the choices of a,b,c,d,e,f and thus this will not always be observable (i.e. in minimal form)!

Summary

Introduced concept of the observability matrix

1. Shown that a system is observable if the matrix M_o is full rank.
2. Given some numerical examples and showed that, if in observable canonical form, a state space matrix is always fully observable.
3. In general computation of M_o and rank testing is tedious, so can be done on a computer.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

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