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State-space analysis 8 detectability and stabilisability

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Introduction

- The previous videos have introduced concepts of observability and controllability.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

- Important questions arise about what statements can be made if a system is not fully observable or not fully controllable.
- Also, one may wish to understand the causes of a lack of observability/controllability and wonder whether certain conditions can guarantee these.

The arguments for the discrete case are exactly analogous so not treated separately.

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Observability/controllability matrix

A convenient test of observability is the rank of M_o.

$$M_O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

A similar test exists for controllability.

$$M_C = [B, AB, A^2B, \cdots, A^{n-1}B]$$

The tests are the same for both continuous and discrete systems.

Using eigenvalue/vector decomposition

A system is fully observable if and only if the **CW** matrix has no zero columns.

$$A = W\Lambda V$$

$$\dot{x} = Ax + Bu;$$
$$y = Cx + Du$$

A system is fully controllable if and only if the matrix **VB** has no zero rows.

The eigenmode approach gives more insight into the concepts of detectability and stabilisability.

Uncontrollable modes

Assume that there exist uncontrollable modes.

$$\left\{\begin{array}{l} \dot{z} = \Lambda x + VBu \\ y = CWz + Du \end{array}\right\}; \quad \begin{bmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_n^T \end{bmatrix} = VB; \quad \beta_j^T = 0$$

Stabilisability means that all unstable modes are controllable.

What happens to the mode $z_j(t)$? Clearly:

$$z_j(t) = e^{\lambda_j t} z_j(0)$$

Stabilisability means that all uncontrollable modes are convergent.

If the eigenvalue is in the RHP, then this mode will diverge to infinity!.

Undectectable modes

Assume that there exist unobservable modes.

$$\left\{\begin{array}{l}\dot{z} = \Lambda x + VBu \\ y = CWz + Du\end{array}\right\}$$

$$A = W\Lambda V; \quad [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n] = CW \quad \gamma_j = 0$$

Dectectability means that all unstable modes are observable.

What impact does the mode $z_j(t)$ have on y(t)? Clearly:

$$z_j(t) = e^{\lambda_j t} z_j(0)$$

If the eigenvalue is in the LHP, then this mode will converge to zero.

Dectectability means that all unobservable modes are convergent.

Remark

1. If a system has full controllability then it is stabilisable.
2. If a system has full observability then it is detectable.

A lack of controllability/observability can be due to

- Cancelling pole/zero pairs.
- A poor choice of output measurement (number and type).
- A poor choice of inputs (number and type).

Remarks

The magnitudes of the variables β_i and Υ_i give an indication of the degree of controllability/ observability of the corresponding modes.

1. Low values mean that these properties are relatively poor so that in practice it may be slow or difficult to gain convergence to the desired values.
2. Poor controllability/reachability can be detected by poor conditionning in the 'M' matrices, that is a M matrix that is close to rank deficient.

Minimal realisations

A system is said to be minimal if it has full controllability and full observability.

1. A non-minimal system has either uncontrollable or unobservable modes or both.
2. Canonical descriptions are not guaranteed both full observable and full controllable, but minimal realisations can be determined from these.
3. A minimal realisation (assuming stabilisability and detectability) is all that is needed for managing input/output behaviour. MATLAB tools exist to give this which removes redundant modes.

NUMERICAL EXAMPLES

Example 1a (Controllable, not observable)

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}; \quad V = W^{-1} = \frac{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}}{3}$$

$$VB = \begin{bmatrix} 0.33 \\ 2.67 \end{bmatrix}; \quad CW = \begin{bmatrix} 3 & 0 \end{bmatrix}$$

$$C(sI - A)^{-1} B = \frac{s+1}{(s+1)(s-2)}$$

Eigenvalue 2 is at -1, therefore in LHP.

DECTECTABLE!

Transfer function has a pole/zero cancellation!

Example 1b (Controllable, not observable)

$$A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix};\quad B=\begin{bmatrix}-2\\ 3\end{bmatrix};\quad C=\begin{bmatrix}1 & -2\end{bmatrix}$$

$$W=\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix};\quad V=W^{-1}=\frac{\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}}{3}$$

$$VB=\begin{bmatrix}0.33\\ 2.67\end{bmatrix};\quad CW=\begin{bmatrix}0 & -3\end{bmatrix}$$

$$C(sI-A)^{-1}B=\frac{-8(s-2)}{(s+1)(s-2)}$$

Eigenvalue 1 is at 2, therefore in RHP.

NOT DECTECTABLE!

Transfer function has a pole/zero cancellation!

Example 2a (Observable, not controllable, not stabilisable)

$$A=\begin{bmatrix}1.8 & 0.6 & -0.2\\ 0.8 & 1.6 & -0.2\\ -0.4 & -0.8 & 2.6\end{bmatrix};\quad B=\begin{bmatrix}1\\1\\0\end{bmatrix};\quad C=\begin{bmatrix}0 & 1 & 2\end{bmatrix}$$

$$VB=\begin{bmatrix}0\\ -1.47\\ -1.33\end{bmatrix};\quad CW=\begin{bmatrix}-1.33 & -2.04 & 1.51\end{bmatrix}$$

$$C(sI-A)^{-1}B=\frac{(s-5)(s-1)}{(s-1)(s-2)(s-3)}$$

Eigenvalue 1 is at 1, therefore in RHP. NOT STABILISABLE!

Transfer function has a pole/zero cancellation!

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Example 2b (Observable, not controllable BUT STABILISABLE)

$$A=\begin{bmatrix}-1.8 & -0.6 & 0.2\\ -0.8 & -1.6 & 0.2\\ 0.4 & 0.8 & -2.6\end{bmatrix};\quad B=\begin{bmatrix}1\\1\\0\end{bmatrix};\quad C=\begin{bmatrix}0 & 1 & 2\end{bmatrix}$$

$$VB=\begin{bmatrix}0\\1.47\\-1.33\end{bmatrix};\quad C=\begin{bmatrix}1.33 & 2.04 & 1.51\end{bmatrix}$$

$$C(sI-A)^{-1}B=\frac{(s+5)(s+1)}{(s+1)(s+2)(s+3)}$$

Eigenvalue 1 is at -1, therefore in LHP. THEREFORE STABILISABLE!

Transfer function has a pole/zero cancellation!

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REMARK

The 4 previous examples all had pole/zero cancellations in the corresponding transfer functions.

They were not in minimal form.

Summary

Introduced concepts of the detectability and stabilisability.

$$\dot{x} = Ax + Bu; \quad y = Cx$$

1. A system that has full observability and controllability is always detectable and stabilisable and in minimal form.
2. A lack of either controllability or observability indicates the system is not in minimal form. Depending on the nature of the uncontrollable/unobservable modes such a system may or may not lack stabilisability/detectability.

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