The University Of Sheffield.

# State-space behaviours 1 introduction

J A Rossiter

# Introduction

- The first chapter demonstrated a number of ways of generating a state space model to represent a linear system.

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- This set of videos focuses on how the behaviours link to the parameters in A,B,C,D. (We will assume D=0 as this is typical with strictly proper systems.)

# Eigenvalues and transfer functions

It was shown in video 1.5 that one can find an equivalent transfer function model from a state space model, to represent input/output behaviour.

$$\dot{x} = Ax + Bu; \qquad y = Cx + Du$$

$$(sI - A)X = BU \quad \Rightarrow \quad \begin{aligned} X &= (sI - A)^{-1}BU \\ Y &= [C(sI - A)^{-1}B + D]U \end{aligned}$$

Roots of |sI-A| correspond to the eigenvalues of A.

Poles come from determinant of (sI-A).

# Interim summary

We expect the modes of behaviour of a state space model to be determined by the eigenvalues of the A matrix.

- This video series will not get side tracked by special cases with embedded pole/zero cancellations, repeated poles, non-simple Jordan forms and the like.
- We will focus on constant inputs u(t) so that analytic solutions are straightforward. In general the dynamics of u(t) will appear in the output.

# Expectation

Given a model and constant u.

$$\dot{x} = Ax + Bu; \quad y = Cx$$

$$eigenvalues \equiv |\lambda I - A| = 0$$

$$y(t) = k + w_1 e^{\lambda_1 t} + w_2 e^{\lambda_2 t} + \cdots + w_n e^{\lambda_n t}$$

How in general might the vectors $w_i$ be determined? (k is simple to evaluate)

# Using Laplace transforms

For now focus on the case with no input and investigate the state behaviour.

$$\dot{x} = Ax$$

$$X(s) = (sI - A)^{-1} x(0)$$

A number of examples will be presented showing that the free response can be determined this way, although it is rather inefficient.

Also implicit that a state transition matrix can be defined so that:

$$x(t) = \Phi(t)x(0); \quad \Phi(t) = L^{-1}[(sI - A)^{-1}]$$

# EXAMPLE 1

$$\Phi(s) = \left(sI - \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}\right)^{-1} = \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix}}{s^2+3s+2}$$

$$\Phi(s) = \begin{bmatrix} \dfrac{s}{s^2+3s+2} & \dfrac{-1}{s^2+3s+2} \\ \dfrac{1}{s^2+3s+2} & \dfrac{s+3}{s^2+3s+2} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{s+1} + \dfrac{2}{s+2} & \dfrac{-2}{s+1} + \dfrac{2}{s+2} \\ \dfrac{1}{s+1} + \dfrac{-1}{s+2} & \dfrac{2}{s+1} + \dfrac{-1}{s+2} \end{bmatrix}$$

$$\Phi(t) = L^{-1}[\Phi(s)] = \begin{bmatrix} -e^{-t} + 2e^{-2t} & -2e^{-t} + 2e^{-2t} \\ e^{-t} - e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

# EXAMPLE 2

$$\Phi(s) = \left(sI - \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)^{-1} = \frac{\begin{bmatrix} s^2 & -11s-6 & -6s \\ s & s^2+6s & -6 \\ 1 & s+6 & s^2+6s+11 \end{bmatrix}}{s^3+6s^2+11s+6}$$

Clearly very tedious and not a route to be pursued in general!

$$\Phi(s) = \begin{bmatrix} \frac{0.5}{s+1}+\frac{-0.25}{s+2}+\frac{4.5}{s+3} & \frac{2.5}{s+1}+\frac{-16}{s+2}+\frac{13.5}{s+3} & \frac{3}{s+1}+\frac{-3}{s+2}+\frac{9}{s+3} \\ \frac{-0.5}{s+1}+\frac{2}{s+2}+\frac{-1.5}{s+3} & \frac{-2.5}{s+1}+\frac{8}{s+2}+\frac{-4.5}{s+3} & \frac{-3}{s+1}+\frac{6}{s+2}+\frac{-3}{s+3} \\ \frac{0.5}{s+1}+\frac{-1}{s+2}+\frac{0.5}{s+3} & \frac{2.5}{s+1}+\frac{-4}{s+2}+\frac{1.5}{s+3} & \frac{3}{s+1}+\frac{-3}{s+2}+\frac{1}{s+3} \end{bmatrix}$$

# State transition matrix Φ(t)

It is demonstrated that:

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0)$$

The state transition matrix Φ(t) can be computed using Laplace methods, although this is tedious.

# Properties of state transition matrix Φ(t)

It is easy to show that Φ(t) has a number of properties, although these are unlikely to be used much hereafter.

$$\Phi(t_1+t_2)=\Phi(t_1)\Phi(t_2)$$

$$\Phi^{-1}(t)=\Phi(-t)$$

$$x(t_1+t_2)=\Phi(t_1+t_2)x(0);$$
$$x(t_1+t_2)=\Phi(t_1)x(t_2)$$
$$x(t_2)=\Phi(t_2)x(0)$$

$$x(t)=\Phi(t)x(0);$$
$$x(0)=\Phi(-t)x(t)$$
$$\left[\Phi(t)\Phi(-t)\right]=I$$

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Anthony Rossiter

Department of Automatic Control and Systems Engineering

University of Sheffield

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