State-space behaviours 2 using eigenvalues

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Introduction

The first video demonstrated that one can solve

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0)$$

- The state transition matrix Φ(t) can be computed using Laplace methods, although this is tedious

$$\Phi(t) = L^{-1}[(sI - A)^{-1}]$$

- This video looks at an alternative derivation using similarity transforms and eigenvalue/vector decompositions.

Interim summary

We expect the modes of behaviour of a state space model to be determined by the eigenvalues of the A matrix.

This video series will not get side tracked by special cases with embedded pole/zero cancellations, repeated poles and non-simple Jordan forms and the like.

$$\Phi(t) = L^{-1}[(sI - A)^{-1}]$$

Poles come from determinant of (sI-A) which are clearly the same as the eigenvalues of A.

1st order example and extension

Consider the case where there is only one state.

In this case the state space model reduces to a standard 1st order differential equation whose behaviours are well understood.

$$\dot{x} = ax \quad \Rightarrow \quad x(t) = e^{at}x(0)$$

Can we derive an equivalent solution for matrices?

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = e^{At}x(0)$$

Remark

From the previous video we know that

$$\dot{x} = Ax \quad \Rightarrow \quad x = \Phi(t)x(0)$$

$$\Phi(t) = L^{-1}\left[(sI - A)^{-1}\right]$$

A simplistic statement could be to define the following as the meaning of matrix exponential.

$$\dot{x} = e^{At}x(0) = \Phi(t)x(0) \quad \Rightarrow \quad e^{At} = \Phi(t)$$

Alternative insight/key result

For now ignore the system input and consider the system dynamics (transition matrix).

$$\dot{x} = Ax \quad \Rightarrow \quad x = e^{At}x(0)$$

This definition of Φ(t) accords well with the rules for differentiation of exponentials of scalars.

$$\frac{d}{dt}e^{At} = Ae^{At}$$

Typical text books use Maclaurin expansions to prove this core result.

Definition of e^{At}

Where it exists (distinct eigenvalues), it may be easier to use an eigenvalue/vector decomposition.

$$\frac{d}{dt} e^{At} = A e^{At}$$

$$\left\{ \begin{matrix} \dot{x} = Ax & A = W\Lambda V \\ x = Wz & z = Vx \end{matrix} \right\} \Rightarrow \quad \dot{z} = VAWz;$$

$$\dot{z} = \Lambda z \quad \Rightarrow \quad z = e^{\Lambda t} z(0) \quad \Rightarrow \quad x = W e^{\Lambda t} V x(0)$$

$$e^{At} = W e^{\Lambda t} V$$

Definition of e^{At}

Take the result from the previous page and note that the middle matrix is diagonal.

$$e^{At} = We^{\Lambda t}V; \quad e^{\Lambda t} = diag[e^{\lambda_1 t}, \cdots, e^{\lambda_n t}]$$

Therefore:

$$\frac{d}{dt}e^{At} = \frac{d}{dt}We^{\Lambda t}V = W\begin{bmatrix} \lambda_1 e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n e^{\lambda_n t} \end{bmatrix}V$$

$$W\Lambda e^{\Lambda t}V = \underbrace{W\Lambda V}_{A}\underbrace{We^{\Lambda t}V}_{e^{At}} = Ae^{At} = \frac{d}{dt}\left(e^{At}\right)$$

State transition matrix e^{At}

It is well accepted that:

$$\dot{x} = Ax \quad \Rightarrow \quad x = e^{At}x(0)$$

The state transition matrix Φ(t) can be defined as follows using an eigenvalue/vector decomposition.

$$e^{At} = \Phi(t) = We^{\Lambda t}V = L^{-1}\left[(sI - A)^{-1}\right]$$

This is useful as it emphasises the role of the eigenvalues in the dynamics of the solution and also exploits scalar computations where that is helpful.

Summary

The behaviours of a state-space system are governed by the eigenvalues of the A matrix.

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = e^{At}x(0)$$

This result follows directly from a Laplace transform analysis and also from a similarity transform using the eigenvectors.

For distinct eigenvalues, the state transition matrix is given as:

$$e^{At} = We^{\Lambda t}V \quad \equiv \quad L^{-1}[(sI - A)^{-1}]$$

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