



State-space behaviours 2 using eigenvalues

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Introduction

The first video demonstrated that one can solve

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0)$$

- The state transition matrix $\Phi(t)$ can be computed using Laplace methods, although this is tedious

$$\Phi(t) = L^{-1}[(sI - A)^{-1}]$$

- This video looks at an alternative derivation using similarity transforms and eigenvalue/vector decompositions.

Interim summary

We expect the modes of behaviour of a state space model to be determined by the eigenvalues of the A matrix.

This video series will not get side tracked by special cases with embedded pole/zero cancellations, repeated poles and non-simple Jordan forms and the like.

$$\Phi(t) = L^{-1}[(sI - A)^{-1}]$$

Poles come from determinant of $(sI - A)$ which are clearly the same as the eigenvalues of A .

1st order example and extension

Consider the case where there is only one state. In this case the state space model reduces to a standard 1st order differential equation whose behaviours are well understood.

$$\dot{x} = ax \quad \Rightarrow \quad x(t) = e^{at} x(0)$$

Can we derive an equivalent solution for matrices?

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = e^{At} x(0)$$

Remark

From the previous video we know that

$$\dot{x} = Ax \quad \Rightarrow \quad x = \Phi(t)x(0)$$

$$\Phi(t) = L^{-1} \left[(sI - A)^{-1} \right]$$

A simplistic statement could be to define the following as the meaning of matrix exponential.

$$\dot{x} = e^{At} x(0) = \Phi(t)x(0) \quad \Rightarrow \quad e^{At} = \Phi(t)$$

Alternative insight/key result

For now ignore the system input and consider the system dynamics (transition matrix).

$$\dot{x} = Ax \quad \Rightarrow \quad x = e^{At} x(0)$$

This definition of $\Phi(t)$ accords well with the rules for differentiation of exponentials of scalars.

$$\frac{d}{dt} e^{At} = A e^{At}$$

Typical text books use Maclaurin expansions to prove this core result.

Definition of e^{At}

Where it exists (distinct eigenvalues), it may be easier to use an eigenvalue/vector decomposition.

$$\frac{d}{dt} e^{At} = A e^{At}$$

$$\left\{ \begin{array}{l} \dot{x} = Ax \quad A = W\Lambda V \\ x = Wz \quad z = Vx \end{array} \right\} \Rightarrow \dot{z} = VAWz;$$

$$\dot{z} = \Lambda z \Rightarrow z = e^{\Lambda t} z(0) \Rightarrow x = W e^{\Lambda t} V x(0)$$

$$e^{At} = W e^{\Lambda t} V$$

Definition of e^{At}

Take the result from the previous page and note that the middle matrix is diagonal.

$$e^{At} = We^{\Lambda t}V; \quad e^{\Lambda t} = \text{diag}[e^{\lambda_1 t}, \dots, e^{\lambda_n t}]$$

Therefore:

$$\frac{d}{dt} e^{At} = \frac{d}{dt} We^{\Lambda t}V = W \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n e^{\lambda_n t} \end{bmatrix} V$$

$$W\Lambda e^{\Lambda t}V = \underbrace{W\Lambda V}_A \underbrace{e^{\Lambda t}V}_{e^{At}} = Ae^{At} = \frac{d}{dt} (e^{At})$$

State transition matrix e^{At}

It is well accepted that:

$$\dot{x} = Ax \quad \Rightarrow \quad x = e^{At} x(0)$$

The state transition matrix $\Phi(t)$ can be defined as follows using an eigenvalue/vector decomposition.

$$e^{At} = \Phi(t) = We^{\Lambda t}V = L^{-1} \left[(sI - A)^{-1} \right]$$

This is useful as it emphasises the role of the eigenvalues in the dynamics of the solution and also exploits scalar computations where that is helpful.

Summary

The behaviours of a state-space system are governed by the eigenvalues of the A matrix.

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = e^{At} x(0)$$

This result follows directly from a Laplace transform analysis and also from a similarity transform using the eigenvectors.

For distinct eigenvalues, the state transition matrix is given as:

$$e^{At} = We^{\Lambda t}V \quad \equiv \quad L^{-1}[(sI - A)^{-1}]$$



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