The University Of Sheffield.

State-space behaviours 3 step response

J A Rossiter

Slides by Anthony Rossiter

Introduction

The previous videos focussed on the solutions for initial conditions and no input.

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0)$$

$$e^{At} = \Phi(t) = We^{\Lambda t}V = L^{-1}\left[(sI - A)^{-1}\right]$$

This video considers what happens when the input u(t) is a constant.

$$\dot{x} = Ax + Bu \quad \Rightarrow \quad x(t) = \Phi(t)x(0) + H(t)u$$

This is the step response

Approach taken

Two obvious possibilities are:

1. Use Laplace transforms
2. Use a convolution integral.

We will mention the first possibility briefly as it was established in video 1 that this technique, in general is cumbersome and difficult.

In fact it is clear that these computations are not paper and pen exercises in general so these derivations are for understanding - **use a software tool to get numerical answers**!

Laplace approach

It is obvious that, for u(t) a constant and ignoring initial conditions:

$$\dot{x} = Ax + Bu$$

$$(sI - A)X = B\frac{u}{s} \quad \Rightarrow \quad X = (sI - A)^{-1}B\frac{u}{s}$$

$$x(t) = H(t)u; \quad H(t) = L^{-1}\left[\frac{(sI - A)^{-1}B}{s}\right]$$

This is likely to be tedious and difficult to use.

CONVOLUTION INTEGRAL METHODS

1^{st} order example and extension

For a system with impulse response e^{at}, a convolution integral can be used to determine the response with a constant input (and indeed a time varying input).

$$\dot{x} = ax + bu \quad \Rightarrow \quad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} bu\,d\tau$$

An equivalent solution for matrices is obvious.

$$\dot{x} = Ax + Bu \quad \Rightarrow \quad x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu\,d\tau$$

Convolution integral

The integral is easy to perform as we know:

$$\frac{d}{dt}e^{At} = Ae^{At}$$

$$\int_0^t e^{A(t-\tau)}Bud\tau = -A^{-1}\left[e^{A(t-\tau)}Bu\right]_0^t$$

$$= -A^{-1}\left[e^{A(0)}Bu\right] - -A^{-1}\left[e^{A(t)}Bu\right]$$

$$H(t) \quad = A^{-1}\left(e^{At} - I\right)B$$

Total response and summary

Using superposition, the step response from non-zero initial conditions is given as:

$$\dot{x} = Ax + Bu \quad \Rightarrow \quad x(t) = \Phi(t)x(0) + H(t)u$$

$$\Phi(t) = e^{At}; \quad H(t) \quad = A^{-1}\left(\Phi(t) - I\right)B$$

With an output equation y=Cx, one can also see:

$$y(t) = C\Phi(t)x(0) + CH(t)u$$

The University Of Sheffield.

Open Engineering Resources

HEA Engineering Subject Centre
A HEFCE/JISC/HEA Pilot Project

Anthony Rossiter
Department of Automatic Control and Systems Engineering
University of Sheffield
www.shef.ac.uk/acse