State-space behaviours 4 eigenmodes

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Introduction

The previous videos focussed on the solutions for initial conditions and a constant input.

$$\dot{x} = Ax + Bu \quad \Rightarrow \quad x(t) = \Phi(t)x(0) + H(t)u$$

$$\Phi(t) = e^{At}; \quad H(t) \quad = A^{-1}\left(\Phi(t) - I\right)B$$

This video considers what insight can be derived from the eigenvalue/vector decomposition.

$$e^{At} = We^{\Lambda t}V$$

Eigenvalue/vector decomposition

The decomposition (distinct eigenvalues) is:

$$A = W\Lambda V = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} = I = VW$$

$$Aw_i = \lambda_i w_i$$

Approach taken

This video will focus only on the impact of non-zero initial conditions and not the step response.

Therefore the focus is on the part of the response given as:

$$x(t) = We^{\Lambda t}Vx(0)$$

Focus here is on the case of distinct eigenvalues only.

Projection onto eigenvectors

The initial condition can be projected onto the eigenvectors.

$$x(0) = \alpha_1 w_1 + \cdots + \alpha_n w_n$$

The relevant coefficients can be determined as follows:

$$\alpha_i = v_i^T x(0)$$

$$x(0) = [w_1 v_1^T + \cdots + w_n v_n^T] x(0) = WVx(0)$$

Superposition

Given a state space model is linear, superposition can be used to determine the overall response.

Here superposition will be applied by considered the projection of x(0) onto each eigenvector.

Hence analyse each component in turn.

$$x(0) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$

$$x(t) = We^{\Lambda t} V x(0)$$

Superposition

Hence analyse each component in turn.

$$x(0) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$

$$\alpha_i = v_i^T x(0)$$

$$\left\{ v_i^T w_j = 0, \quad i \neq j \right\} \quad \left\{ v_i^T w_i = 1 \right\}$$

$$z_1 = We^{\Lambda t} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \alpha_1 w_1 = We^{\Lambda t} \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = w_1 e^{\lambda_1 t} \alpha_1$$

Superposition continued

Using the same solution for each component.

$$x(0) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$

$$\alpha_i = v_i^T x(0)$$

$$x(t) = z_1 + z_2 + \cdots + z_n$$

$$z_1 = w_1 e^{\lambda_1 t} \alpha_1; \quad \cdots \quad ; z_n = w_n e^{\lambda_n t} \alpha_n$$

Analyse solution

The solution has n distinct modes linked directly to the eigenvalues.

$$x(t) = w_1 e^{\lambda_1 t} \alpha_1 + \cdots + w_n e^{\lambda_n t} \alpha_n$$

The contribution, or decay, along each eigenvector direction is linked directly to the corresponding eigenvalue.

An illustration of this is best done through several examples.

NUMERICAL EXAMPLES

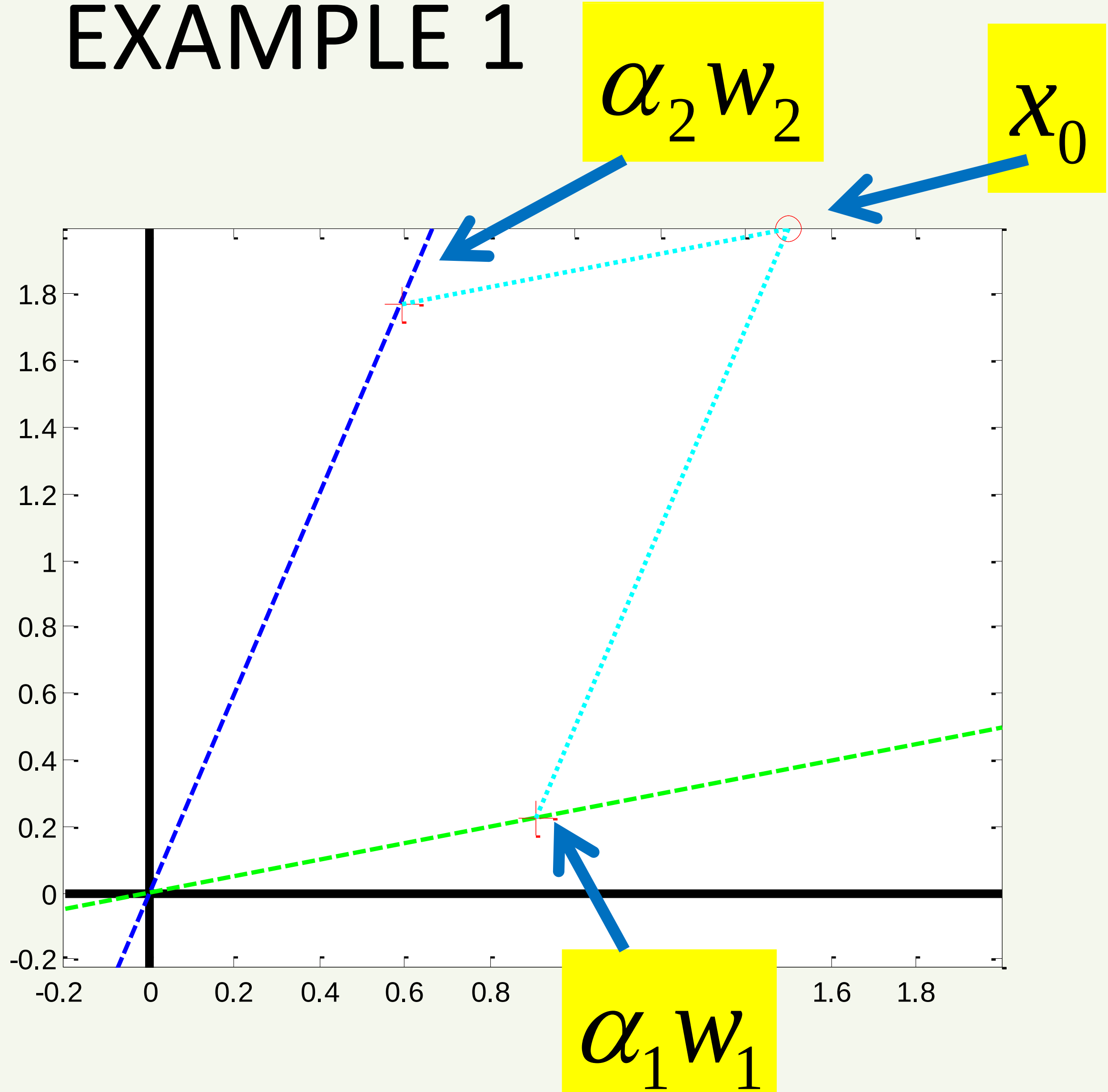
EXAMPLE 1

$$A = \begin{bmatrix} -2.09 & 0.36 \\ -0.27 & -0.91 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 0.33 \\ 0.25 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

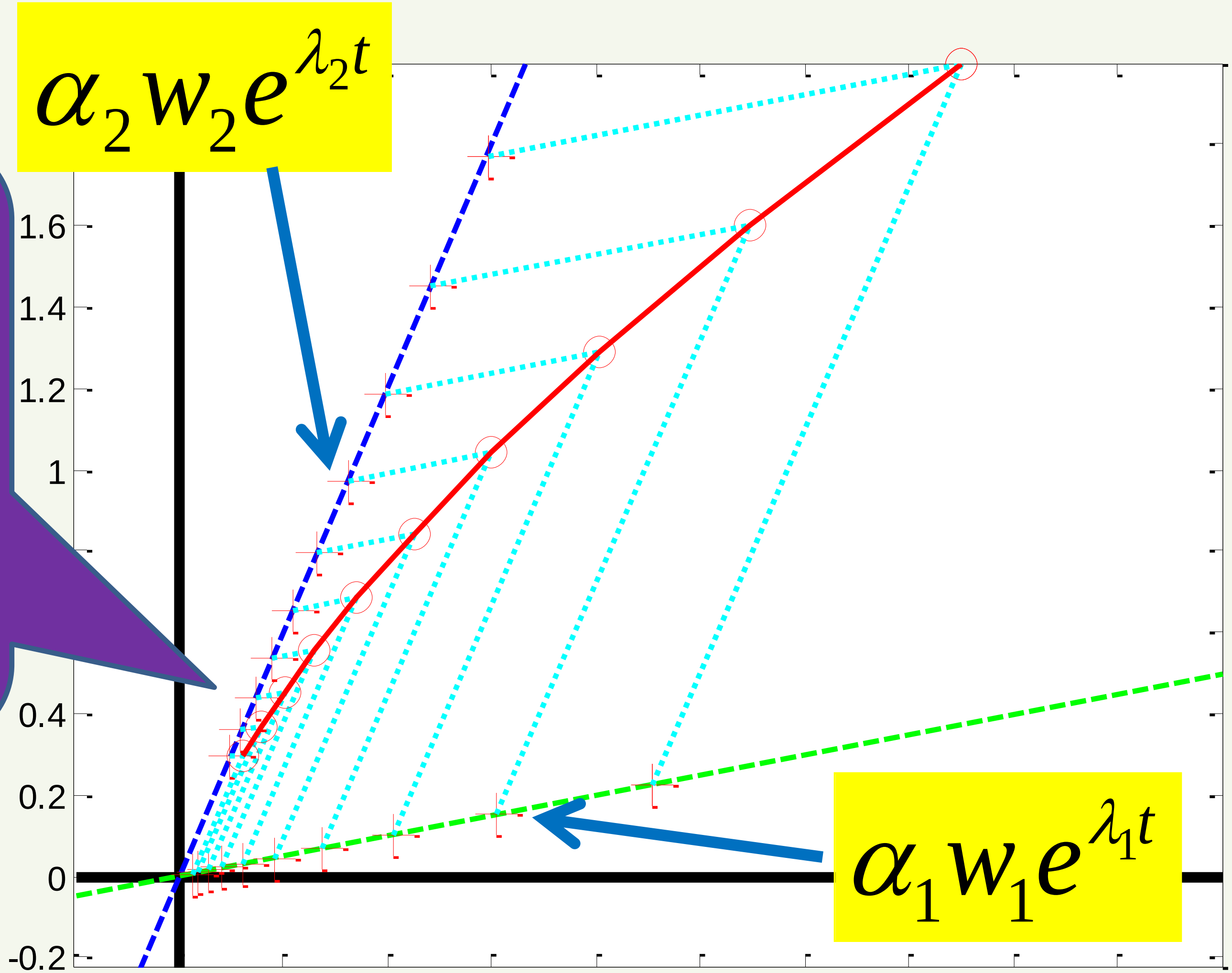
$$x(0) = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$$



Next, see what happens as we run forward in time. This is best done live on MATLAB (phaseplane.m)

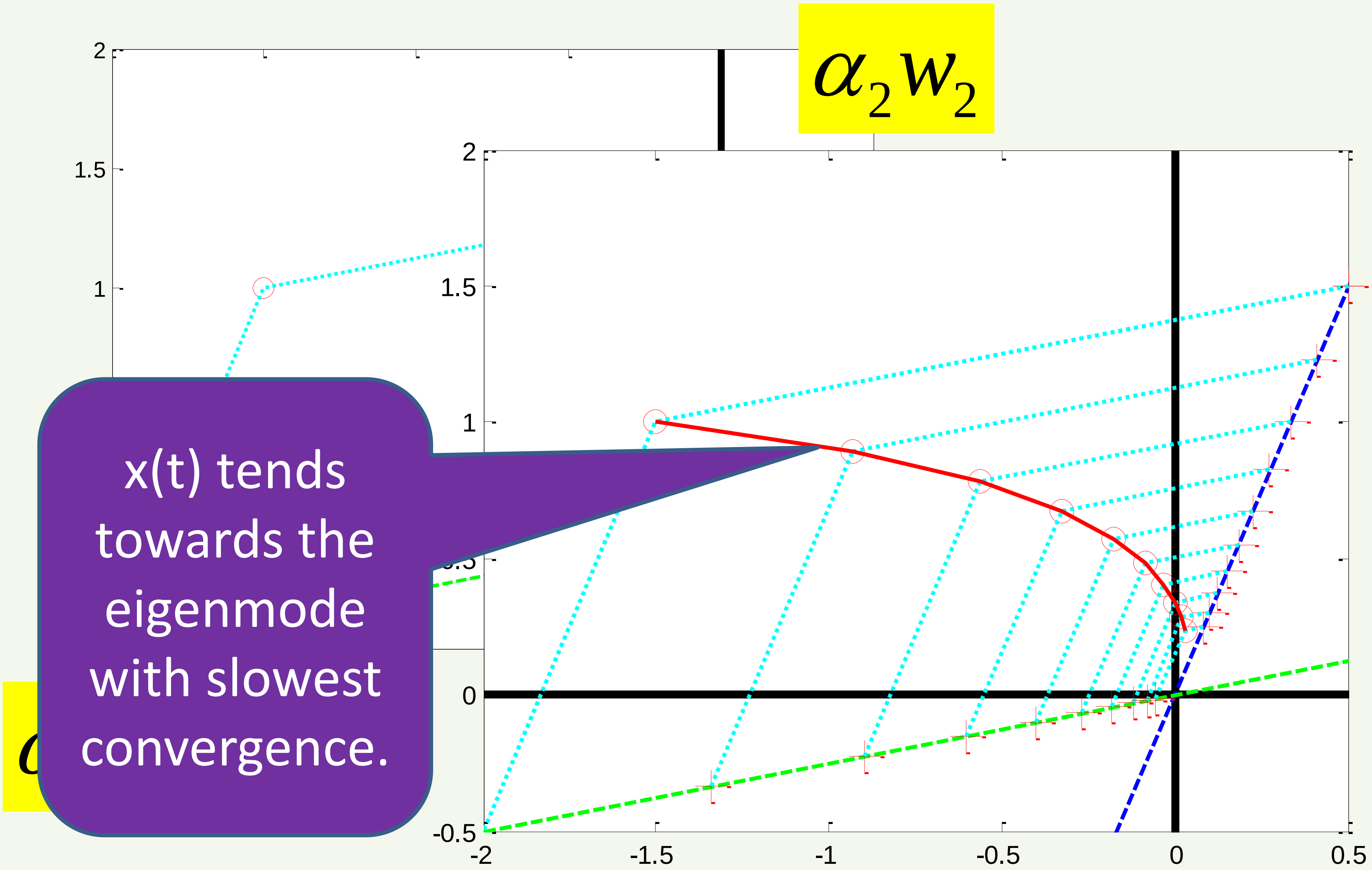
Example 1 continued

x(t) tends towards the eigenmode with slowest convergence.



$$\left\{ t \to \infty, e^{\lambda_2 t} \succ e^{\lambda_1 t} \right\} \Rightarrow \left| \alpha_2 w_2 e^{\lambda_2 t} \right| \succ \left| \alpha_1 w_1 e^{\lambda_1 t} \right|$$

Alternative initial condition



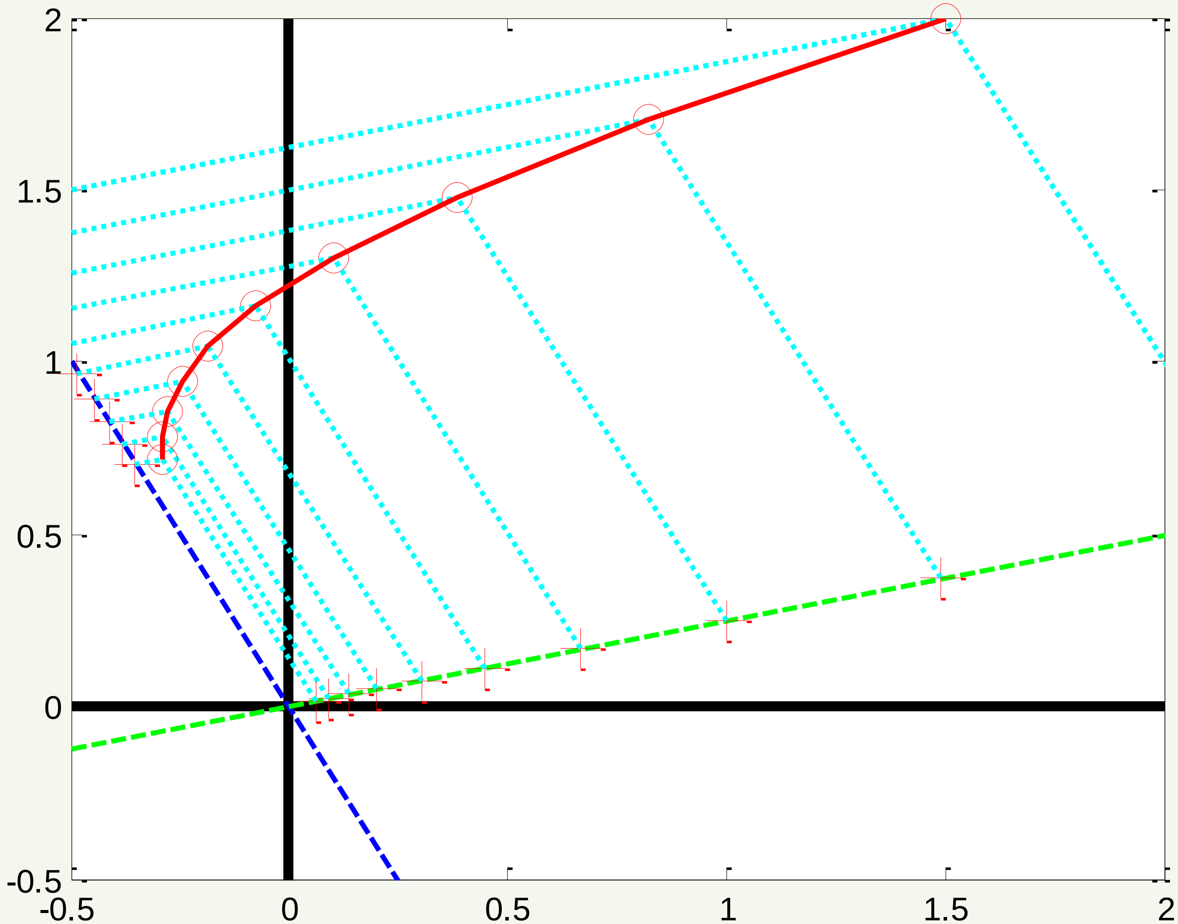
EXAMPLE 2

$$A = \begin{bmatrix} -1.82 & -0.71 \\ -0.36 & -0.58 \end{bmatrix}$$

$$W = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -0.4 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$$



x(t) tends towards the eigenvector with slowest convergence.

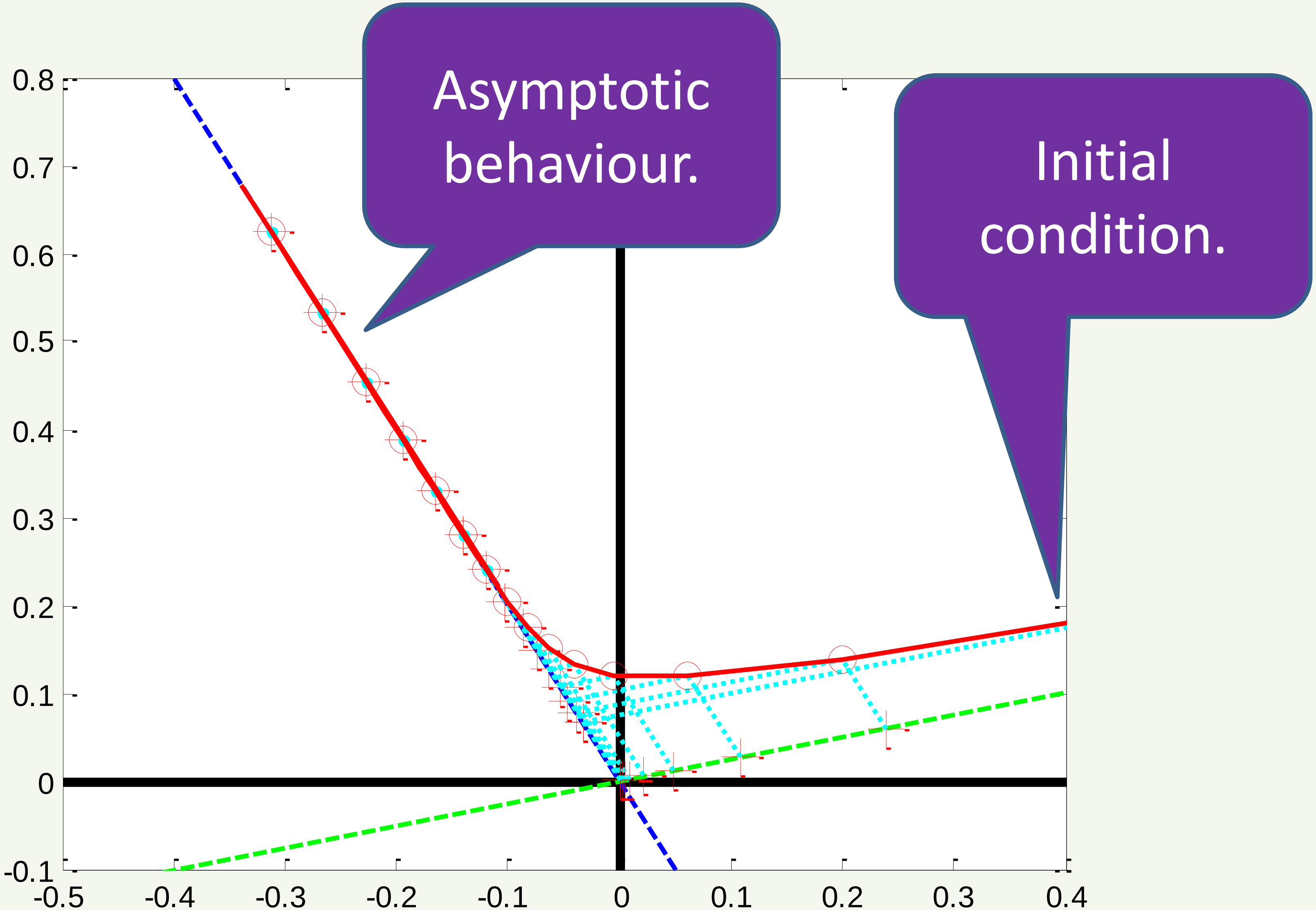
Remark

If just one of the eigenvalues corresponds to a divergent mode then clearly the trajectory will approach this asymptotically.

$$x(t) = w_1 e^{\lambda_1 t} \alpha_1 + \cdots + w_n e^{\lambda_n t} \alpha_n$$

$$\mathrm{Re}(\lambda_i) > 0 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = w_i e^{\lambda_i t} \alpha_i$$

Example with unstable mode



Analyse solution

The solution of a state space model has n distinct modes linked directly to the eigenvalues.

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0); \quad \Phi(t) = e^{At}$$

$$x(t) = w_1 e^{\lambda_1 t}\alpha_1 + \cdots + w_n e^{\lambda_n t}\alpha_n \qquad \alpha_i = v_i^T x(0)$$

- The contribution, or decay, along each eigenvector direction is linked directly to the corresponding eigenvalue.
- So far we have considered real and distinct eigenvalues only.

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