



State-space behaviours 4 eigenmodes

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Introduction

The previous videos focussed on the solutions for initial conditions and a constant input.

$$\dot{x} = Ax + Bu \quad \Rightarrow \quad x(t) = \Phi(t)x(0) + H(t)u$$

$$\Phi(t) = e^{At}; \quad H(t) = A^{-1}(\Phi(t) - I)B$$

This video considers what insight can be derived from the eigenvalue/vector decomposition.

$$e^{At} = We^{\Lambda t}V$$

Eigenvalue/vector decomposition

The decomposition (distinct eigenvalues) is:

$$A = W\Lambda V = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} = I = VW$$

$$Aw_i = \lambda_i w_i$$

Approach taken

This video will focus only on the impact of non-zero initial conditions and not the step response.

Therefore the focus is on the part of the response given as:

$$x(t) = We^{\Lambda t}Vx(0)$$

Focus here is on the case of distinct eigenvalues only.

Projection onto eigenvectors

The initial condition can be projected onto the eigenvectors.

$$x(0) = \alpha_1 w_1 + \cdots + \alpha_n w_n$$

The relevant coefficients can be determined as follows:

$$\alpha_i = v_i^T x(0)$$

$$x(0) = [w_1 v_1^T + \cdots + w_n v_n^T] x(0) = W V x(0)$$

Superposition

Given a state space model is linear, superposition can be used to determine the overall response.

Here superposition will be applied by considered the projection of $x(0)$ onto each eigenvector.

Hence analyse each component in turn.

$$x(0) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$

$$x(t) = W e^{\Lambda t} V x(0)$$

Superposition

Hence analyse each component in turn.

$$x(0) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$

$$\alpha_i = v_i^T x(0)$$

$$\left\{ v_i^T w_j = 0, \quad i \neq j \right\} \quad \left\{ v_i^T w_i = 1 \right\}$$

$$z_1 = W e^{\Lambda t} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \alpha_1 w_1 = W e^{\Lambda t} \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = w_1 e^{\lambda_1 t} \alpha_1$$

Superposition continued

Using the same solution for each component.

$$x(0) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$

$$\alpha_i = v_i^T x(0)$$

$$x(t) = z_1 + z_2 + \cdots + z_n$$

$$z_1 = w_1 e^{\lambda_1 t} \alpha_1; \quad \cdots \quad ; z_n = w_n e^{\lambda_n t} \alpha_n$$

Analyse solution

The solution has n distinct modes linked directly to the eigenvalues.

$$x(t) = w_1 e^{\lambda_1 t} \alpha_1 + \cdots + w_n e^{\lambda_n t} \alpha_n$$

The contribution, or decay, along each eigenvector direction is linked directly to the corresponding eigenvalue.

An illustration of this is best done through several examples.

NUMERICAL EXAMPLES

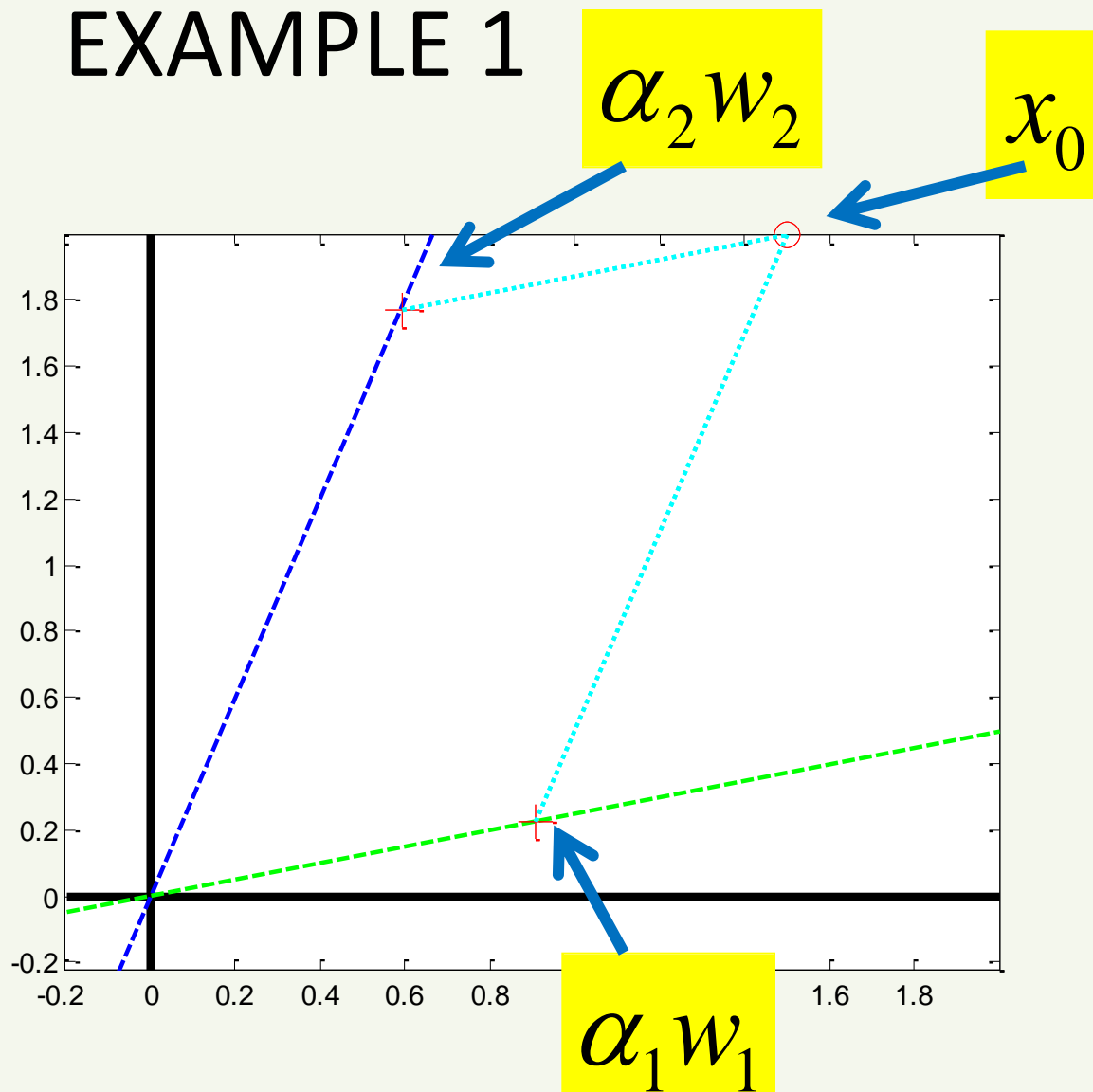
EXAMPLE 1

$$A = \begin{bmatrix} -2.09 & 0.36 \\ -0.27 & -0.91 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 0.33 \\ 0.25 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

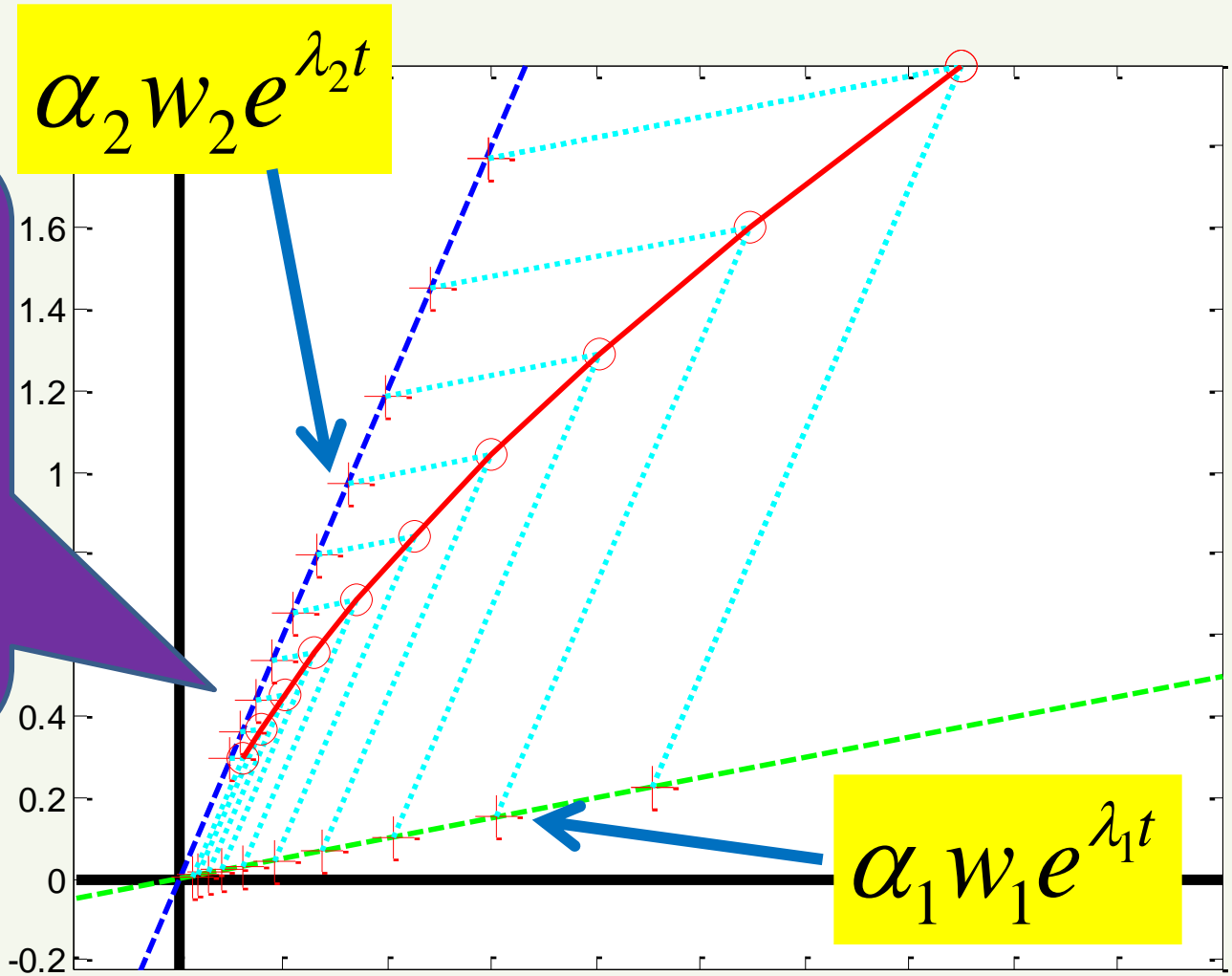
$$x(0) = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$$



Next, see what happens as we run forward in time. This is best done live on MATLAB (phaseplane.m)

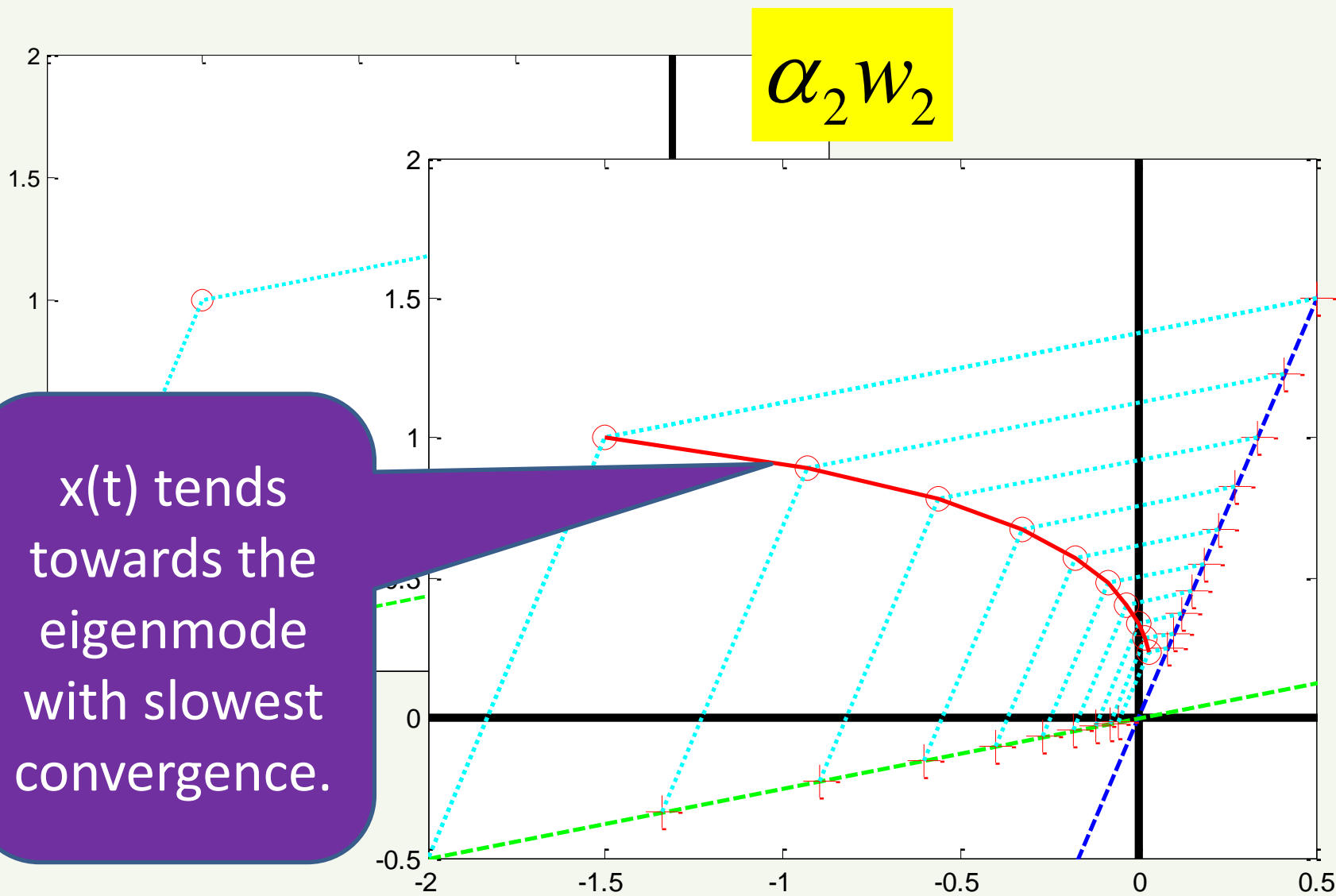
Example 1 continued

$x(t)$ tends towards the eigenmode with slowest convergence.



$$\left\{ t \rightarrow \infty, e^{\lambda_2 t} \succ e^{\lambda_1 t} \right\} \Rightarrow \left| \alpha_2 w_2 e^{\lambda_2 t} \right| \succ \left| \alpha_1 w_1 e^{\lambda_1 t} \right|$$

Alternative initial condition



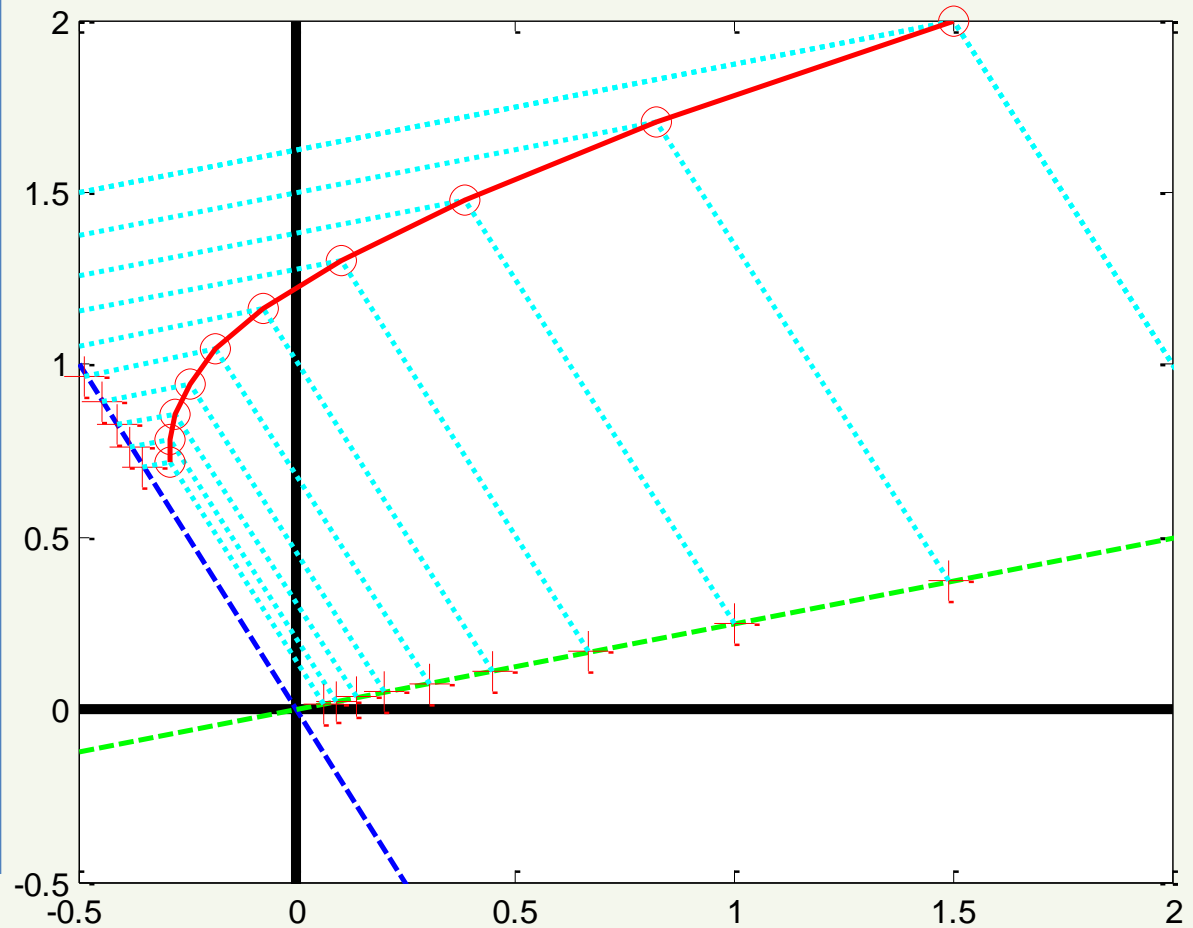
EXAMPLE 2

$$A = \begin{bmatrix} -1.82 & -0.71 \\ -0.36 & -0.58 \end{bmatrix}$$

$$W = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -0.4 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$$



$x(t)$ tends towards the eigenvector with slowest convergence.

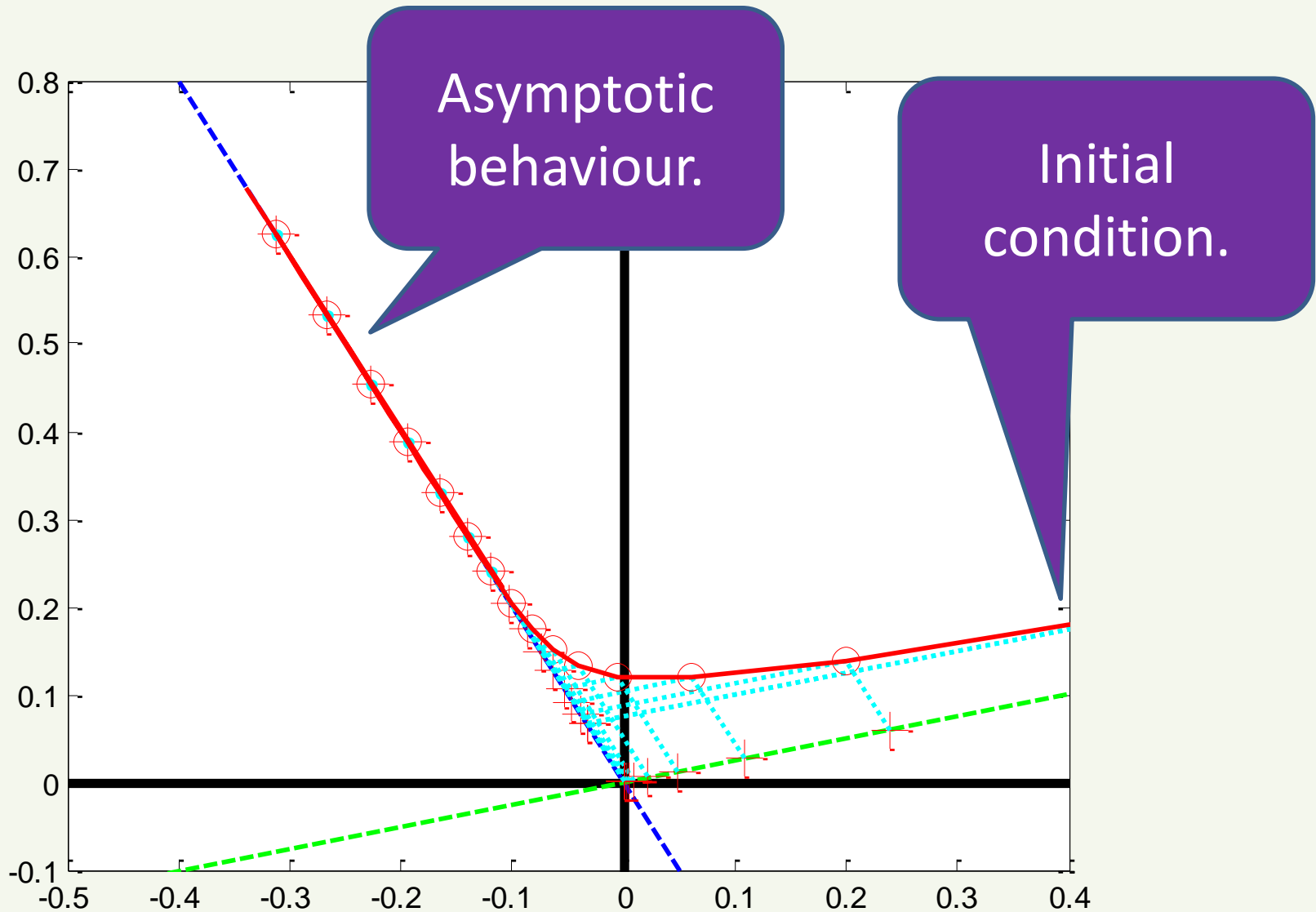
Remark

If just one of the eigenvalues corresponds to a divergent mode then clearly the trajectory will approach this asymptotically.

$$x(t) = w_1 e^{\lambda_1 t} \alpha_1 + \cdots + w_n e^{\lambda_n t} \alpha_n$$

$$\operatorname{Re}(\lambda_i) > 0 \quad \Rightarrow \quad \lim_{t \rightarrow \infty} x(t) = w_i e^{\lambda_i t} \alpha_i$$

Example with unstable mode



Analyse solution

The solution of a state space model has n distinct modes linked directly to the eigenvalues.

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0); \quad \Phi(t) = e^{At}$$

$$x(t) = w_1 e^{\lambda_1 t} \alpha_1 + \cdots + w_n e^{\lambda_n t} \alpha_n \quad \alpha_i = v_i^T x(0)$$

- The contribution, or decay, along each eigenvector direction is linked directly to the corresponding eigenvalue.
- So far we have considered real and distinct eigenvalues only.



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