State-space behaviours 5 oscillatory behaviours

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Introduction

Video 4 demonstrated phase plane plots for free responses of state space models using insights from an eigenvalue/vector decompositions.

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0); \quad \Phi(t) = e^{At}$$

$$e^{At} = We^{\Lambda t}V$$

$$x(t) = w_1 e^{\lambda_1 t}\alpha_1 + \cdots + w_n e^{\lambda_n t}\alpha_n$$

$$\alpha_i = v_i^T x(0)$$

This video considers what changes when the eigenvalues are complex, but still distinct.

Approach taken

This video will focus only on the impact of a single pair of complex eigenvalues.

It should be possible to generalise for multiple pairs, but as meaningful sketches must largely be in 2 dimensions this would be harder to illustrate.

One can use identical analysis to the previous video but simply use complex numbers rather than real numbers, recognising that x(t) must always be real.

Eigenvalue/vector decomposition

The decomposition (distinct eigenvalues)is:

$$A = W\Lambda V = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

Complex eigenvalues have complex eigenvectors which also are complex conjugates of each other!

$$\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \vdots \\ v_n^T \end{bmatrix} = I = VW$$

$$Aw_i = \lambda_i w_i$$

Assume some eigenvalues are complex.

Eigenvalue/vector decomposition

The decomposition is:

$$A = W\Lambda V = \begin{bmatrix} w_1 & \overline{w_1} \end{bmatrix} \begin{bmatrix} a+jb & 0 \\ 0 & a-jb \end{bmatrix} \begin{bmatrix} v_1^T \\ \overline{v_1^T} \end{bmatrix}$$

$$x(t) = w_1 e^{\lambda_1 t}\alpha_1 + \cdots + w_n e^{\lambda_n t}\alpha_n$$

$$x(t) = X_1 e^{\lambda_1 t} + \overline{X_1} e^{\overline{\lambda_1} t}; \quad \lambda_1 = a + ib$$

$$x(t) = e^{at}[Z\cos bt + Q\sin bt]$$

Z,Q depend upon initial conditions.

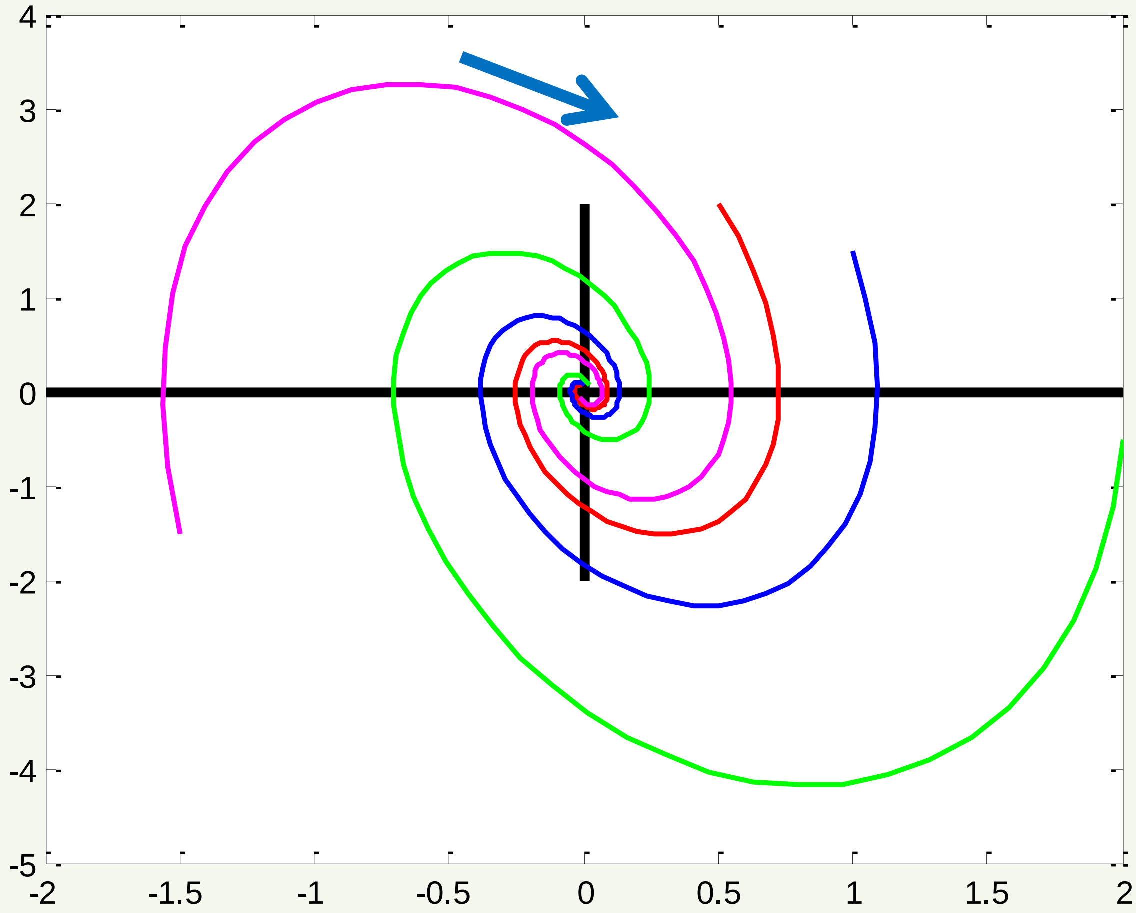
NUMERICAL EXAMPLES

The University Of Sheffield.

EXAMPLE 1

$$A = \begin{bmatrix} 0 & 0.1 \\ -1 & -0.2 \end{bmatrix}$$

$$\lambda = -0.1 \pm j0.3$$



See how the states spiral into the origin due to the oscillatory mode.

Remarks

The oscillation, especially for very slow decays will have a major and minor axis as with ellipses.

However, it is beyond the remit of this set of videos to analyse the scenario in that level of detail.

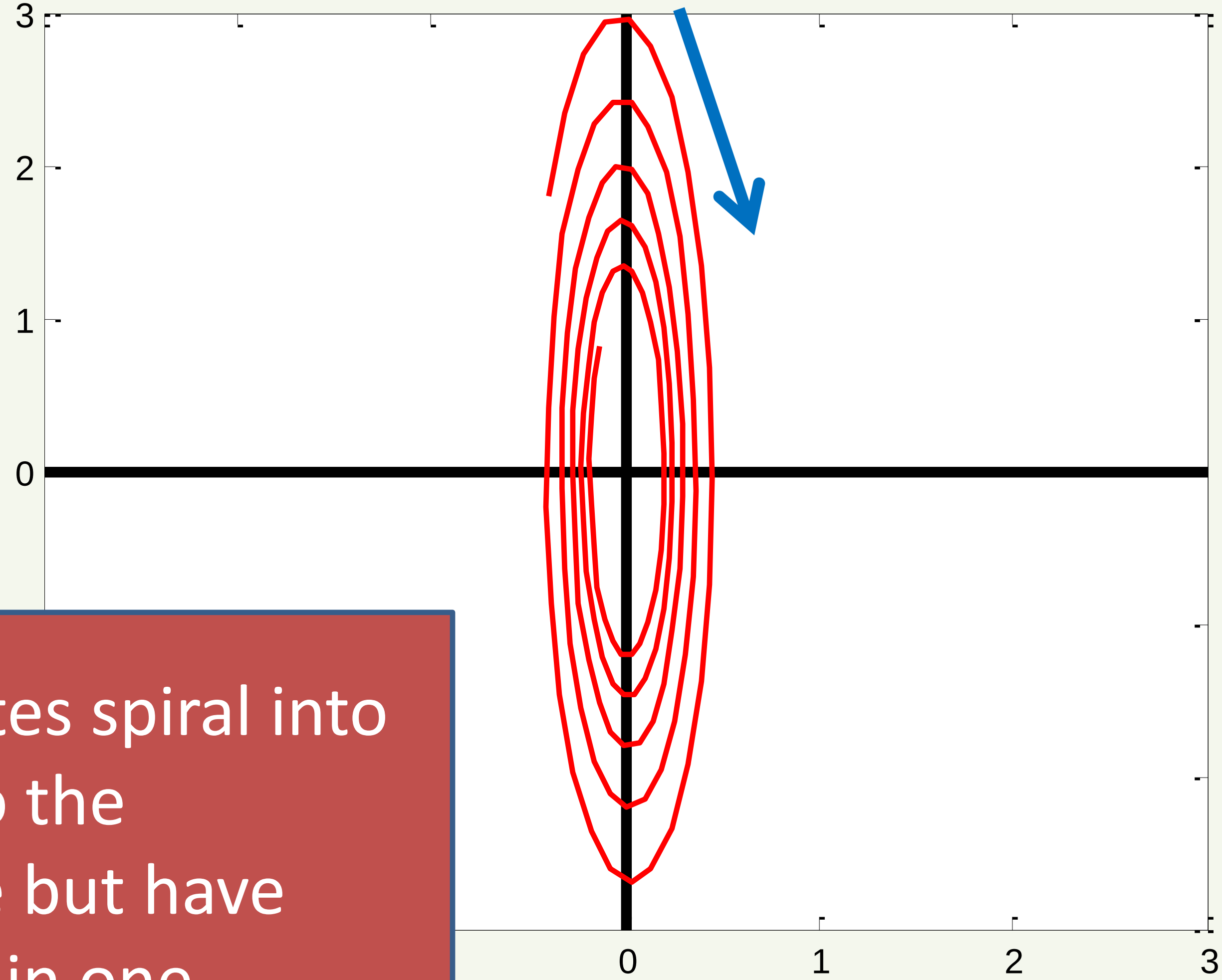
$$x(t) = e^{at}[Z\cos bt + Q\sin bt]$$

Z, Q depend upon initial conditions but clearly may have very different magnitudes and are 90° out of phase.

EXAMPLE 2

$$A = \begin{bmatrix} 0 & 0.1 \\ -4 & -0.04 \end{bmatrix}$$

$$\lambda = -0.02 \pm j0.63$$



See how the states spiral into the origin due to the oscillatory mode but have more oscillation in one direction than another.

Summary

- Complex eigenvalues imply the presence of oscillatory modes.
- This results in spiral like trajectories in the phase plane.
- Depending on system characteristics such as the damping ratio, this spiralling may have obvious major and minor axis, as with an ellipse.
- In general computation is not a paper and pen exercise.

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