



# State-space behaviours 5 oscillatory behaviours

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# Introduction

Video 4 demonstrated phase plane plots for free responses of state space models using insights from an eigenvalue/vector decompositions.

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0); \quad \Phi(t) = e^{At}$$

$$e^{At} = We^{\Lambda t}V$$

$$x(t) = w_1 e^{\lambda_1 t} \alpha_1 + \cdots + w_n e^{\lambda_n t} \alpha_n \quad \alpha_i = v_i^T x(0)$$

This video considers what changes when the eigenvalues are complex, but still distinct.

# Approach taken

This video will focus only on the impact of a single pair of complex eigenvalues.

It should be possible to generalise for multiple pairs, but as meaningful sketches must largely be in 2 dimensions this would be harder to illustrate.

One can use identical analysis to the previous video but simply use complex numbers rather than real numbers, recognising that  $x(t)$  must always be real.

# Eigenvalue/vector decomposition

The decomposition (distinct eigenvalues) is:

$$A = W\Lambda V = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

Complex eigenvalues have complex eigenvectors which also are complex conjugates of each other!

$$Aw_i = \lambda_i w_i$$

$$\begin{bmatrix} w_1 & \cdots & w_n \\ \vdots \\ v_n^T \end{bmatrix} = I = VW$$

Assume some eigenvalues are complex.

# Eigenvalue/vector decomposition

The decomposition is:

$$A = W\Lambda V = \begin{bmatrix} w_1 & \dots \\ \dots & w_n \end{bmatrix} \begin{bmatrix} a + jb & 0 \\ 0 & a - jb \end{bmatrix} \begin{bmatrix} v_1^T \\ \dots \\ v_n^T \end{bmatrix}$$

$$x(t) = w_1 e^{\lambda_1 t} \alpha_1 + \dots + w_n e^{\lambda_n t} \alpha_n$$

$$x(t) = X_1 e^{\lambda_1 t} + \overline{X_1} e^{\overline{\lambda_1} t}; \quad \lambda_1 = a + ib$$

$$x(t) = e^{at} [Z \cos bt + Q \sin bt]$$

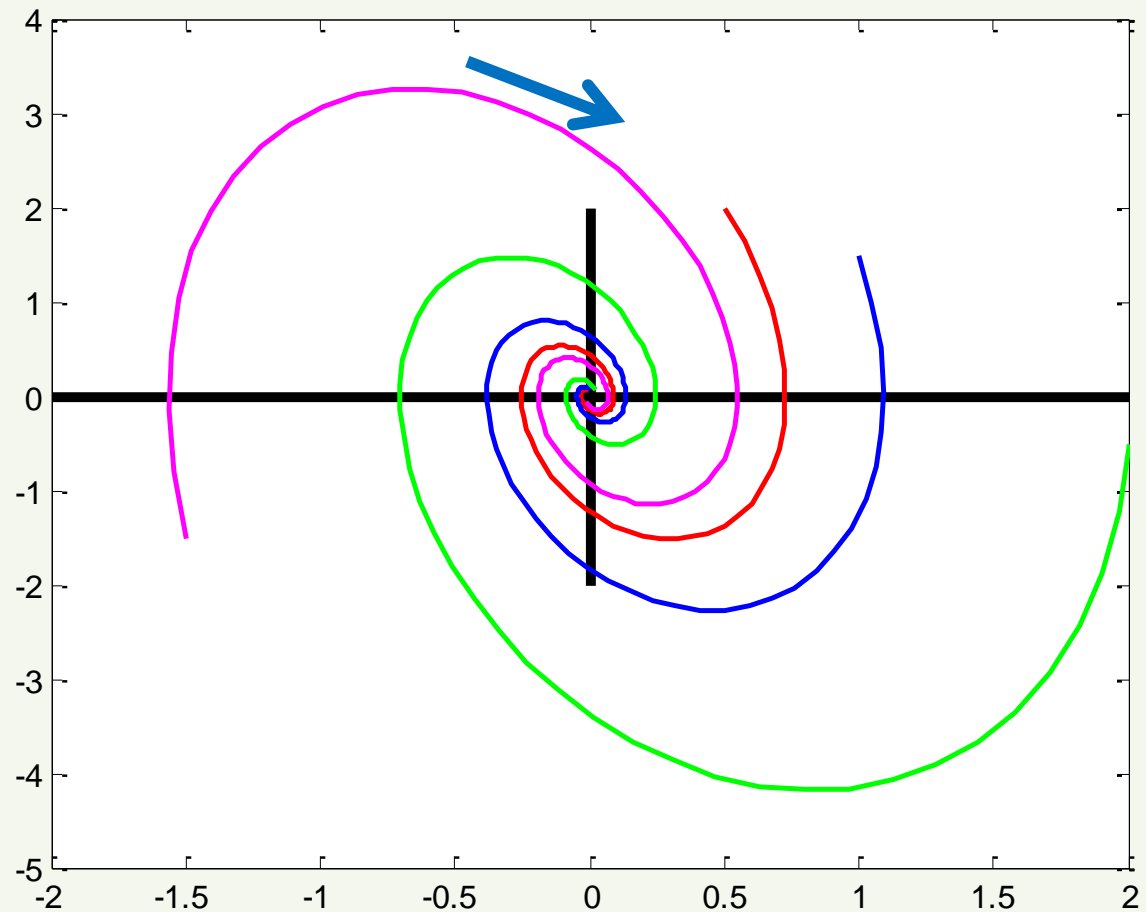
Z, Q depend upon initial conditions.

# NUMERICAL EXAMPLES

# EXAMPLE 1

$$A = \begin{bmatrix} 0 & 0.1 \\ -1 & -0.2 \end{bmatrix}$$

$$\lambda = -0.1 \pm j0.3$$



See how the states spiral into the origin due to the oscillatory mode.

# Remarks

The oscillation, especially for very slow decays will have a major and minor axis as with ellipses.

However, it is beyond the remit of this set of videos to analyse the scenario in that level of detail.

$$x(t) = e^{at} [Z \cos bt + Q \sin bt]$$

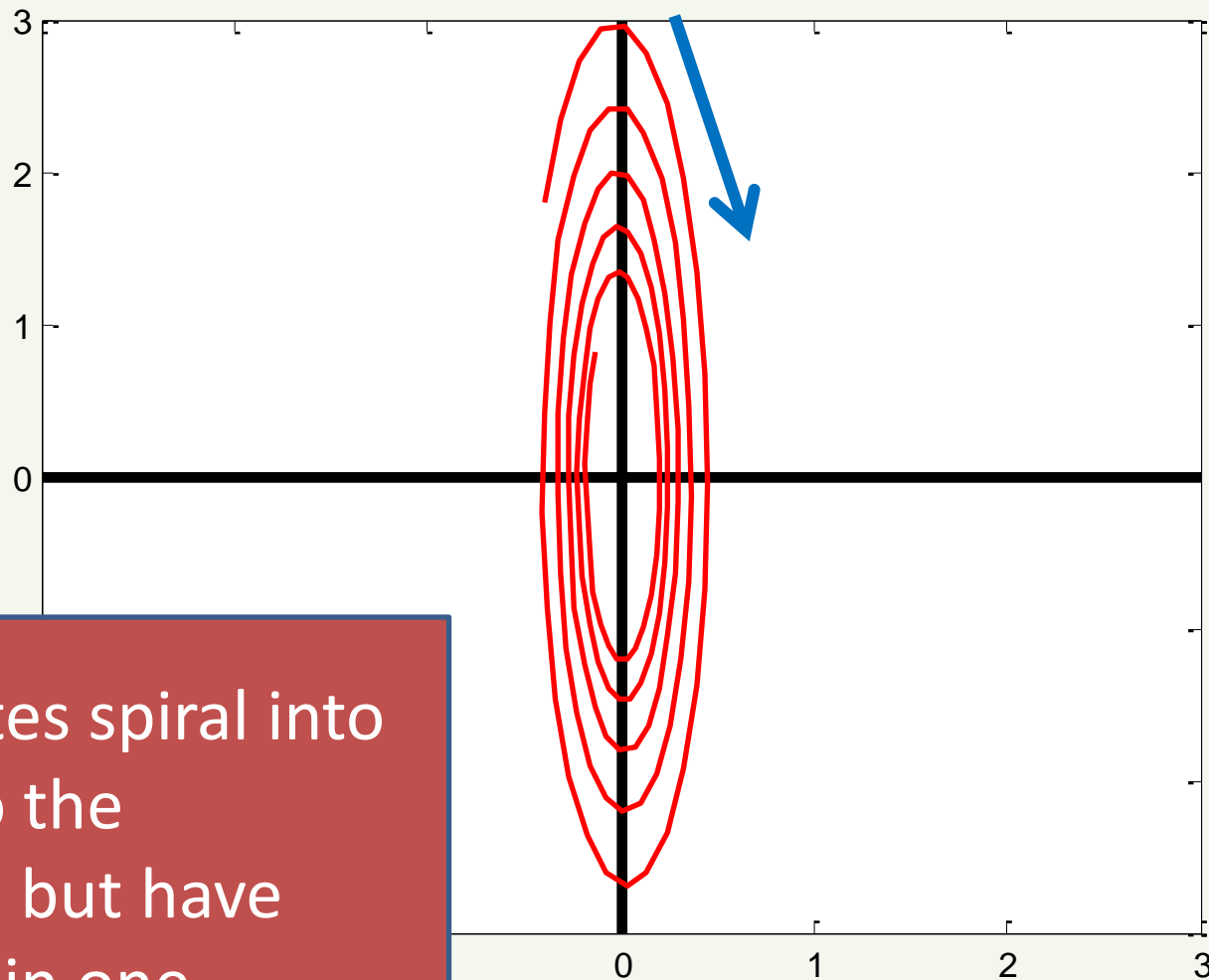
Z, Q depend upon initial conditions but clearly may have very different magnitudes and are 90° out of phase.



# EXAMPLE 2

$$A = \begin{bmatrix} 0 & 0.1 \\ -4 & -0.04 \end{bmatrix}$$

$$\lambda = -0.02 \pm j0.63$$



See how the states spiral into the origin due to the oscillatory mode but have more oscillation in one direction than another.

# Summary

- Complex eigenvalues imply the presence of oscillatory modes.
- This results in spiral like trajectories in the phase plane.
- Depending on system characteristics such as the damping ratio, this spiralling may have obvious major and minor axis, as with an ellipse.
- In general computation is not a paper and pen exercise.



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