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State-space behaviours 6 Cayley-Hamilton Theory

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Introduction

We have demonstrated free responses of state space models using insights from eigenvalue/vector decompositions.

$$\dot{x} = Ax \quad \Rightarrow \quad x(t) = \Phi(t)x(0); \quad \Phi(t) = e^{At}$$

$$e^{At} = We^{\Lambda t}V$$

$$x(t) = w_1 e^{\lambda_1 t}\alpha_1 + \cdots + w_n e^{\lambda_n t}\alpha_n$$

$$\alpha_i = v_i^T x(0)$$

Next we develop some theory which gives insights useful for control design (changing behaviours).

Cayley-Hamilton Theorem

A matrix satisfies its own characteristic equation.

$$|\lambda I - A| = 0 = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

This implies that.

$$A^n + a_{n-1}A^{n-1} + \cdots + a_0 A^0 = 0$$

The proof is omitted but relatively straightforward and available in most text books.

Numerical example

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^2 + 3\lambda + 2$$

$$\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}^2 + 3\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Numerical example

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 6 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^2 - 9\lambda + 26$$

$$\begin{bmatrix} 3 & 4 \\ -2 & 6 \end{bmatrix}^2 - 9\begin{bmatrix} 3 & 4 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 26 & 0 \\ 0 & 26 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 36 \\ -18 & 28 \end{bmatrix} + \begin{bmatrix} -27 & -36 \\ 18 & -54 \end{bmatrix} + \begin{bmatrix} 26 & 0 \\ 0 & 26 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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