State-space behaviours 7 discrete systems

J A Rossiter

Slides by Anthony Rossiter

Introduction

The previous videos focussed on the solutions for continuous time systems.

$$\dot{x} = Ax + Bu \quad \Rightarrow \quad x(t) = \Phi(t)x(0) + H(t)u$$

$$\Phi(t) = e^{At}; \quad H(t) \quad = A^{-1}\left(\Phi(t) - I\right)B$$

$$e^{At} = We^{\Lambda t}V$$

This video considers how these observations change for discrete systems. The observations are given without detailed derivation as mostly analogous.

Discrete state space model

The basic equations we seek take the following form.

$$x_{k+1} = Ax_k + Bu_k \quad \Rightarrow \quad x_{k+n} = \Phi(n)x_k + H(n)u_k$$

This video considers the derivation of the transition matrix Φ(n) and step response matrix H(n).

It is assumed that for a step response:

$$u_k = u_{k+1} = u_{k+2} = \cdots$$

State transition matrix

Equivalent to continuous time we can use transforms and hence:

$$x_{k+1} = Ax_k \quad \Rightarrow \quad (zI - A)x(z) = x_0$$

$$\Phi(z) = \left[(zI - A)^{-1}\right] = I + Az^{-1} + A^2 z^{-2} + \cdots$$

$$x(z) = \Phi(z)x_0 \quad \Rightarrow \quad x_k = A^k x_0$$

$$\Phi(k) = A^k$$

This is also obvious from a recursion of the formula

$$x_{k+1} = Ax_k$$

Remark

In discrete time the state transition matrix has an easier form that in continuous time.

$$\Phi(k) = A^k$$

Nevertheless, computation of this for several values of k is not an insignificant task.

Eigenvalue/vector decomposition

The decomposition (distinct eigenvalues)is:

$$A = W\Lambda V = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} = I = VW$$

$$Aw_i = \lambda_i w_i$$

Discrete system with eigenvalues

For a discrete system, the eigenvalue/vector decomposition can be deployed, as in continuous time, to investigate the impact of different modes on the overall behaviour.

$$x_k = A^k x_0 = W\Lambda^k V x_0$$

Focus here is on the case of distinct eigenvalues only.

Derivation done quickly as equivalent to that in video 5.

Projection onto eigenvectors

The initial condition can be projected onto the eigenvectors.

$$x_0 = \alpha_1 w_1 + \cdots + \alpha_n w_n$$

The relevant coefficients can be determined as follows:

$$\alpha_i = v_i^T x_0$$

Superposition

Hence analyse each component in turn.

$$x_0 = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n$$

$$\alpha_i = v_i^T x_0$$

$$\{v_i^T w_j = 0, \quad i \neq j\} \quad \{v_i^T w_i = 1\}$$

$$x_k = W\Lambda^k V x_0 = w_1 \lambda_1^k \alpha_1 + w_2 \lambda_2^k \alpha_2 + \cdots$$

- The solution has n distinct modes linked directly to the eigenvalues.
- The contribution, or decay, along each eigenvector direction is linked directly to the corresponding eigenvalue.

Step response

The derivation of the step response matrix can be done quickly using the existing results.

$$\left.\begin{array}{r} x_{k+1} = Ax_k + Bu_k \\ u_k = u_{k+1} = u_{k+2} = \cdots \end{array}\right\} \quad \Rightarrow \quad x_{k+n} = \Phi(n)x_k + H(n)u_k$$

$$(zI - A)x(z) = Bu(z) \quad \Rightarrow \quad x(z) = (zI - A)^{-1}Bu(z)$$

$$(zI - A)^{-1}B = Bz^{-1} + ABz^{-2} + A^2Bz^{-3} + \cdots$$

$$x_{k+n} = Bu_{k+n-1} + ABu_{k+n-2} + \cdots + A^{n-1}Bu_k$$

$$u_k = u_{k+1} = \cdots \quad \Rightarrow \quad H(n) = \sum_{i=0}^{n-1} A^i B$$

NUMERICAL EXAMPLES

EXAMPLE 1

$$A = \begin{bmatrix} 0.89 & 0.044 \\ 0.022 & 0.81 \end{bmatrix}$$

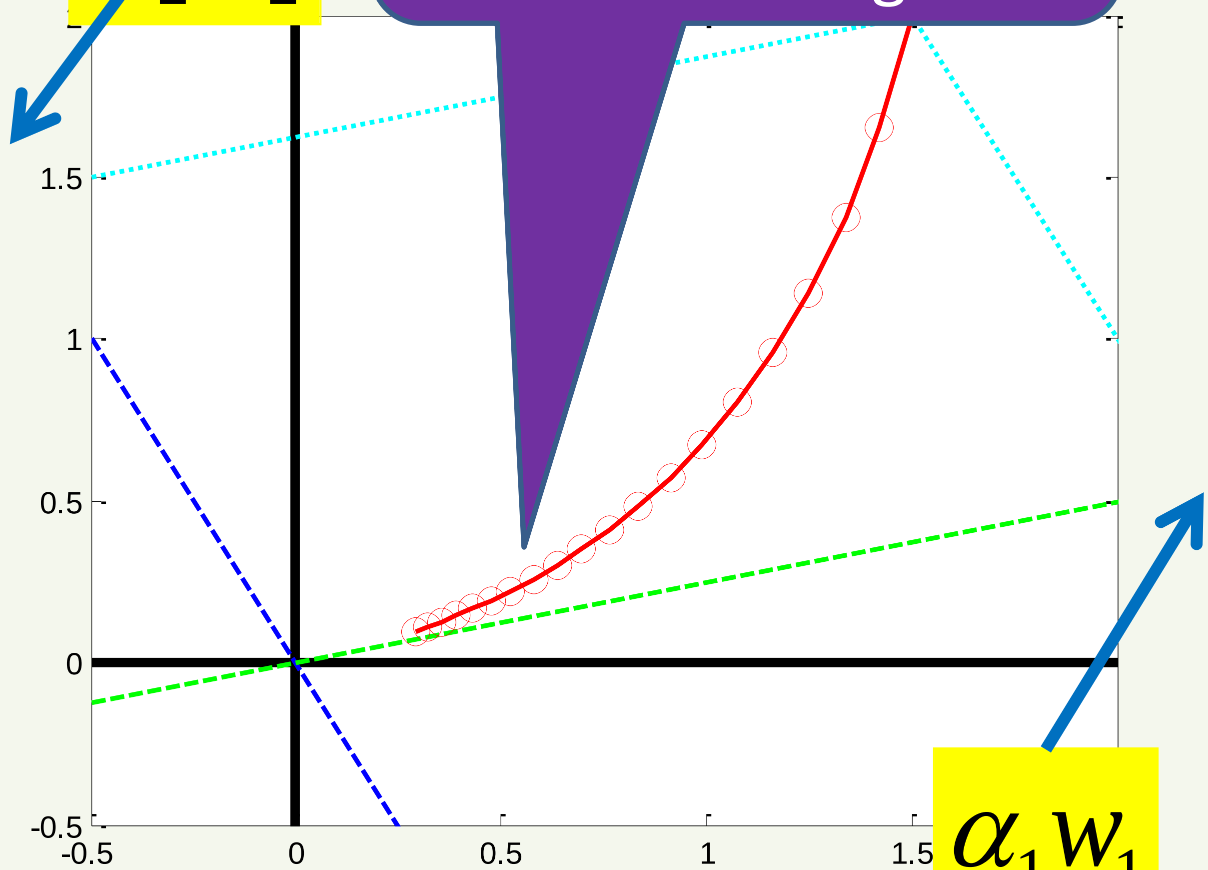
$$W = \begin{bmatrix} 1 & -0.5 \\ 0.25 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$$

$\alpha_2 w_2$

x(k) tends towards the eigenmode with slowest convergence.



$\alpha_1 w_1$

Some concept as observed for continuous time so no more examples are presented (see video 5).

Remark

If just one of the eigenvalues corresponds to a divergent mode then clearly the trajectory will approach this asymptotically.

$$x_k = w_1 \lambda_1^k \alpha_1 + \cdots + w_n \lambda_n^k \alpha_n$$

$$|\lambda_i| > 1 \quad \Rightarrow \quad \lim_{k\to\infty} x_k = w_i \lambda_i^k \alpha_i$$

Summary

The behaviours of discrete models have close analogies to those of the continuous state space models.

Free response:

$$x_k = A^k x_0 = W\Lambda^k V x_0 = w_1 \lambda_1^k v_1^T x_0 + w_2 \lambda_2^k v_2^T x_0 + \cdots$$

Step response.

$$\left. \begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ u_k = u_{k+1} &= u_{k+2} = \cdots \end{aligned} \right\} \Rightarrow \quad x_{k+n} = \Phi(n)x_k + H(n)u_k$$

$$\Phi(n) = A^n; \qquad H(n) = B + AB + \cdots + A^{n-1}B$$

Anthony Rossiter
Department of Automatic Control and Systems Engineering
University of Sheffield
www.shef.ac.uk/acse