State-space models 1
Introduction

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Introduction

• These resources look at a state-space models, their origins, properties and use in control.

• The first few resources focus on origins.

1. What is a state space model?

2. How do I derive a model to represent a given system?

3. Linearised models.

Viewers may find it useful to review the sections on 1\textsuperscript{st} and 2\textsuperscript{nd} order modelling.
State space models

- State space models are defined in terms of so-called system states.
- States are properties which change with time such as speed, temperature, pressure and so forth.
- The model is defined in terms of the derivatives of the states. If you know the derivatives of all the states, then you can capture the system behaviour.
Mass-damper

A simple example of a mass-damper could be a vehicle. The engine provides a drive force whereas, for example, wind and road provide friction or drag.

How do we model such a scenario? (Slower discussion available in modelling sections.)
Modelling a mass-damper

Force balance can now be used to determine the overall model of behaviour.

\[
\begin{align*}
f_1 &= \hat{B}v \\
f_2 &= M \frac{dv}{dt} \\
f &= f_1 + f_2
\end{align*}
\]

This form of mass-damper gives a simple first order differential equation.
Defining state derivatives

The mass-damper can be thought of as having a single state, that is the velocity.

A state space model re-arranges the equation to define the derivative of the state.

This derivative is expressed with matrix notation

$$\begin{align*}
M \frac{dv}{dt} + \hat{B}v &= f \\
\Rightarrow \quad \frac{dv}{dt} &= -\frac{\hat{B}}{M}v + \frac{1}{M}f
\end{align*}$$

State derivative is linear in the state and the input.
Resistor-capacitor in series

Consider the following circuit and use Kirchhoff’s voltage law to derive an appropriate model.

\[
\begin{align*}
\nu_1 &= iR_1 = R_1 \frac{dq}{dt} \\
\nu_2 &= \frac{1}{C} q \\
\nu &= \nu_1 + \nu_2
\end{align*}
\]

\[
\Rightarrow \quad R_1 \frac{dq}{dt} + = \nu
\]
Defining state derivatives

The resistor-capacitor can be thought of as having a single state, that is the charge across the capacitor. A state space model re-arranges the equation to define the derivative of the state.

\[
R \frac{dq}{dt} + \frac{1}{C} q = v \quad \Rightarrow \quad \frac{dq}{dt} = -\frac{1}{RC} q + \frac{1}{R} v
\]

This derivative is expressed with matrix notation

\[
\frac{dq}{dt} = \begin{bmatrix} -\frac{1}{RC} \\ \frac{1}{R} \end{bmatrix} q + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v \quad \Rightarrow \quad \dot{q} = Aq + Bv
\]

State derivative is linear in the state and the input.
Tank with outlet and some inflow

Consider an inflow $F_{in}$ and update the model for the change in depth.

\[
\hat{A} \frac{dh}{dt} = F_{in} - F_{out} \\
K_p F_{out} = \rho gh
\]

\[
\Rightarrow \quad \hat{A} \frac{dh}{dt} = F_{in} - \frac{\rho gh}{K_p}
\]

\[
\frac{dh}{dt} = \left[ -\frac{\rho g}{\hat{A}K_p} \right] h + \left[ \frac{1}{\hat{A}} \right] F_{in} \quad \Rightarrow \quad \dot{h} = Ah + BF_{in}
\]

State derivative is linear in the state and the input.
Summary

Illustrated state-space model derivation for systems that can be modelling a simple 1\textsuperscript{st} order ODE. This amounts to a minor rearrangement and expressing the model as an equation which defines the state derivative with \textbf{linear dependence on a state and an input}.

\[ T \frac{dx}{dt} + x = Ku \quad \Rightarrow \quad \dot{x} = \begin{bmatrix} 1 \\ K \\ T \end{bmatrix} x + \begin{bmatrix} -1 \\ K \\ T \end{bmatrix} u \]

\[ \dot{x} = Ax + Bu \]

It is common to use matrix names $A, B$ for the coefficients of the state and input respectively.
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