State-space models 2
2nd order ODEs

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Introduction

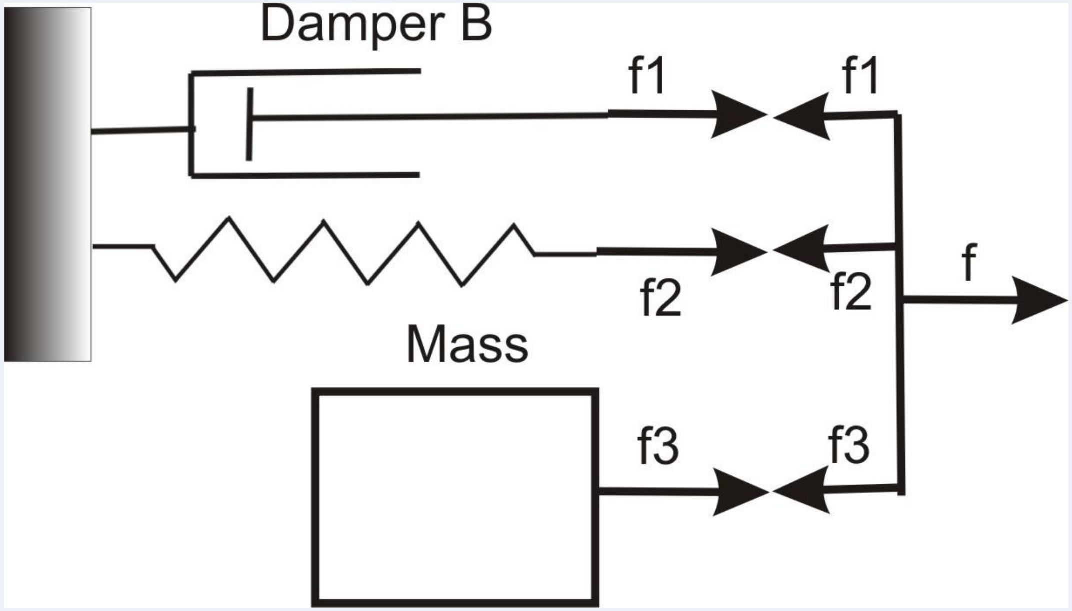
- This resource focuses on derivations of state space model equivalents for systems described by ODEs.
- Here we consider 2nd order ODEs (see separate resources for detailed derivation).
- The state-space model is defined in terms of the derivatives of the states. If you know the derivatives of all the states, then you can capture the system behaviour.
- States relate to dynamic variables such as displacement, height, tension, temperature, etc.

States of a 2nd order ODE

1. The first step in forming a state space model is to define the states.
2. There should be enough independent states to capture the entire system dynamics – for low order systems this selection is usually obvious.
3. However, when presented with a high order differential equation, the user may have no access (or information) relating to the definition of the original underlying states, and thus an arbitrary definition can be used.

Modelling a mass-spring-damper

Force balance can be used to determine the overall model of behaviour.



$$f1 = \hat{B}v$$
$$f2 = kx$$
$$f3 = M\frac{dv}{dt}$$
$$f = f1 + f2 + f3$$

$$f = M\frac{dv}{dt} + \hat{B}v + kx$$

States are v and x

States for mass-spring-damper

$$f = M\frac{dv}{dt} + \hat{B}v + kx$$

It is clear that the model contains states

velocity v and displacement x

so these are a logical choice.

For a state space model, find the derivatives of each state and stack into a single vector.

State-space model

Write the derivatives of v and x and stack in a vector.

$$M\frac{dv}{dt} = f - kx - \hat{B}v \qquad \frac{dx}{dt} = v \quad \Rightarrow \quad \frac{dv}{dt} = \frac{f - kx - \hat{B}v}{M} \qquad \frac{dx}{dt} = v$$

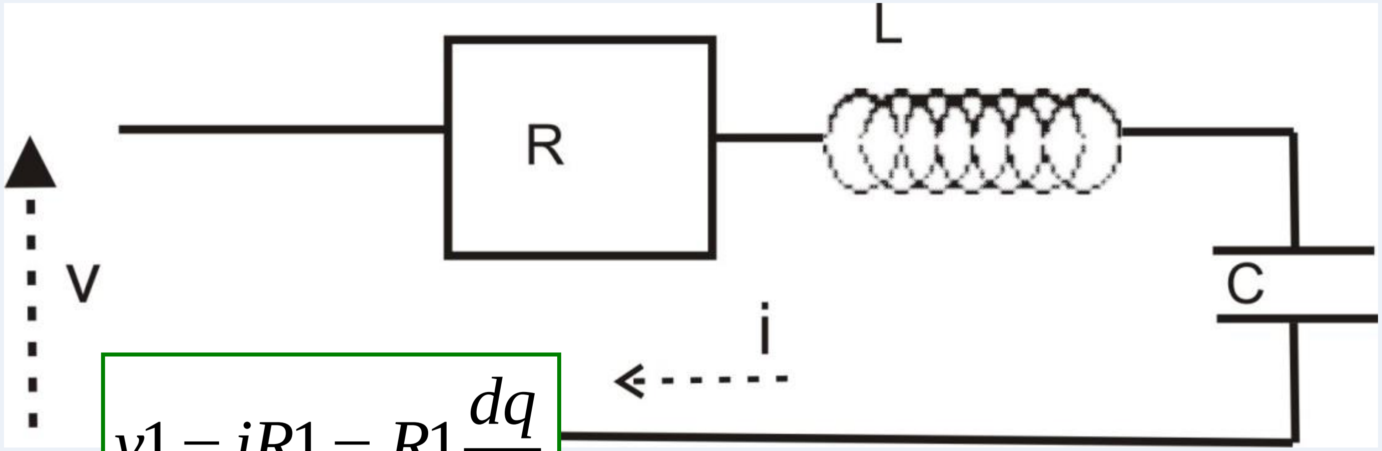
$$\begin{bmatrix} \frac{dv}{dt} \\ \frac{dx}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-\hat{B}}{M} & \frac{-k}{M} \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix}}_{B} f$$

In compact form

$$\dot{z} = Az + Bf$$

Resistor-inductor-capacitor in series

Consider the following circuit and use Kirchhoff's voltage law to derive an appropriate model.



$$v1 = iR1 = R1\frac{dq}{dt}$$

$$v2 = L\frac{di}{dt}$$

$$v3 = \frac{1}{C}q$$

$$v = v1 + v2 + v3$$

$$v = L\frac{di}{dt} + R\frac{dq}{dt} + \left(\frac{1}{C}\right)q$$

States are i and q

State-space model

Write the derivatives of i and q and stack in a vector.

$$L\frac{di}{dt}=v-\frac{q}{C}-iR \qquad \Rightarrow \qquad \frac{di}{dt}=\frac{v-\dfrac{q}{C}-iR}{L}$$

$$\frac{dq}{dt}=i \qquad\qquad \frac{di}{dt}=q$$

$$\begin{bmatrix}\dfrac{di}{dt}\\ \dfrac{dq}{dt}\end{bmatrix}=\underbrace{\begin{bmatrix}\dfrac{-R}{L} & \dfrac{-1}{LC}\\ 1 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}i\\ q\end{bmatrix}}_{z}+\underbrace{\begin{bmatrix}\dfrac{1}{L}\\ 0\end{bmatrix}}_{B}v$$

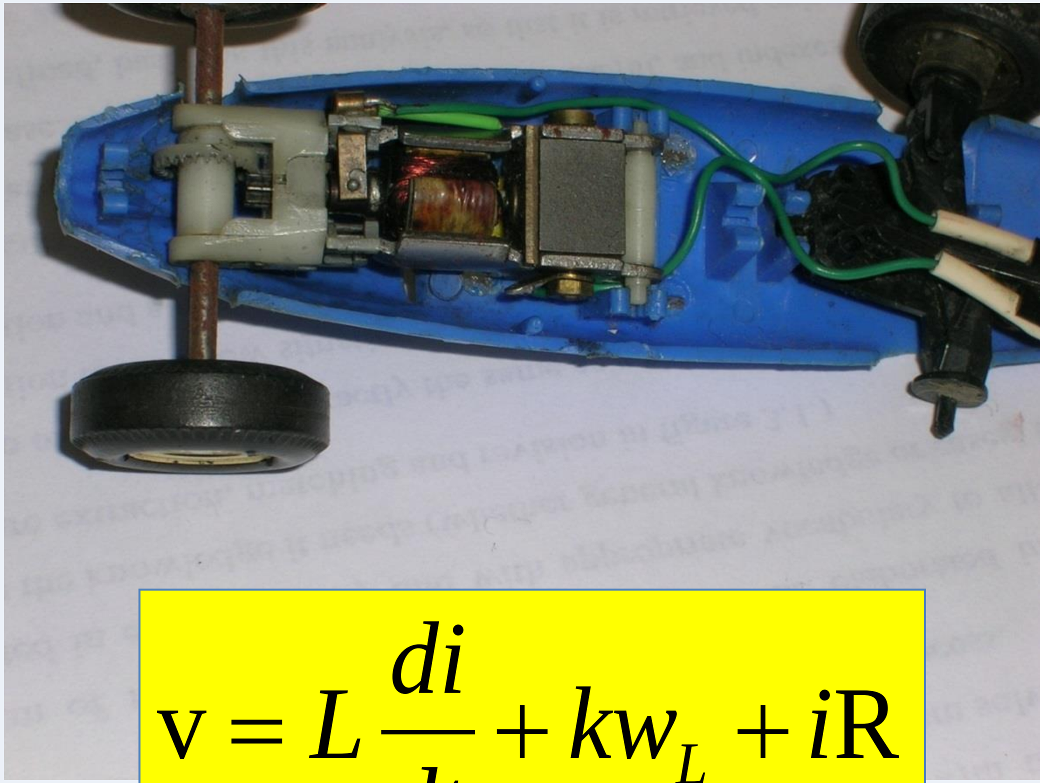
In compact form

$$\dot{z}=Az+Bv$$

DC servo

The model can be summarised with some simple equations.

Here it may be less obvious which states to choice due to the possibilities of two angles/velocities, current, back emf and torque.



$$\mathrm{v} = L\frac{di}{dt} + kw_L + i\mathrm{R}$$

$$ki = \hat{B}w_L + J\frac{dw_L}{dt}$$

DC servo choice of state

The states of interest must be defined with an equation including their derivative.

Other states could be viewed as outputs (possible measurements) and in this case will be linearly dependent on the selected states.

Two obvious choices are w_L and i:

$$\mathrm{v} = L\frac{di}{dt} + kw_L + i\mathrm{R}; \quad ki = \hat{B}w_L + J\frac{dw_L}{dt}$$

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State-space model of DC servo

Express the derivatives of the selected states in a column vector.

$$v = L\frac{di}{dt} + kw_L + i\mathrm{R}; \quad ki = \hat{B}w_L + J\frac{dw_L}{dt}$$

$$\frac{di}{dt} = \frac{v}{L} - \frac{k}{L}w_L - i\frac{\mathrm{R}}{L}; \quad \frac{dw_L}{dt} = \frac{k}{J}i - \frac{\hat{B}}{J}w_L$$

$$\begin{bmatrix} \frac{dw_L}{dt} \\ \frac{di}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{\hat{B}}{J} & \frac{k}{J} \\ -\frac{k}{L} & -\frac{R}{L} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_L \\ i \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{B} v$$

$$\dot{z} = Az + Bv$$

DC servo with displacement

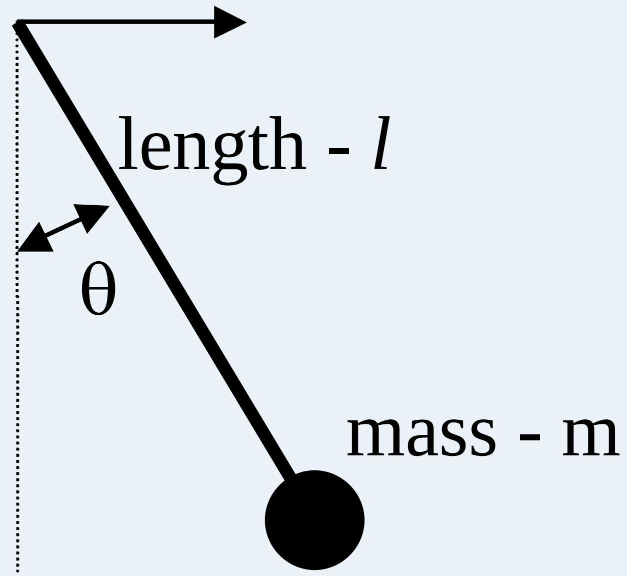
Should the user be interested in the angular position as well as the angular velocity, then an additional state is needed.

$$\mathrm{v} = L\frac{di}{dt} + kw_L + i\mathrm{R}; \quad ki = \hat{B}w_L + J\frac{dw_L}{dt}; \quad w_L = \frac{d\theta_L}{dt}$$

$$\begin{bmatrix} \frac{dw_L}{dt} \\ \frac{di}{dt} \\ \frac{d\theta_L}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{\hat{B}}{J} & \frac{k}{J} & 0 \\ -\frac{k}{L} & -\frac{R}{L} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_L \\ i \\ \theta_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \\ 0 \end{bmatrix} v$$

Pendulum

- A pendulum of length l with end mass m is able to swing freely (assume some friction – constant k).
- A model can be derived using force balance in the tangential direction (small angles).



$$ml\ddot{\theta} = -mg\theta - kl\dot{\theta}$$

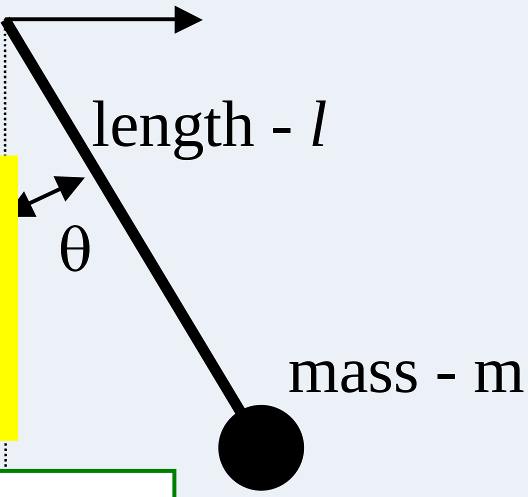
- Choose states:

$$\theta, \quad \dot{\theta} = w$$

Pendulum state space model

State derivatives:

$$ml\frac{dw}{dt} = -mg\theta - klw \qquad \frac{d\theta}{dt} = w$$



$$\begin{bmatrix} \frac{dw}{dt} \\ \frac{d\theta}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-kl}{ml} & \frac{-mg}{ml} \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w \\ \theta \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{B} f$$

$$\dot{z} = Az + Bf$$

Summary

Illustrated state-space model derivation for several common 2nd order systems.

It is noted that the selection of states is important, but this choice is often obvious from the underlying component and balance equations as only certain states will have their derivatives defined explicitly.

The compact form has vectors of states/inputs and matrices of coefficients.

$$\dot{x} = Ax + Bu$$

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